Yo-Yo Variants

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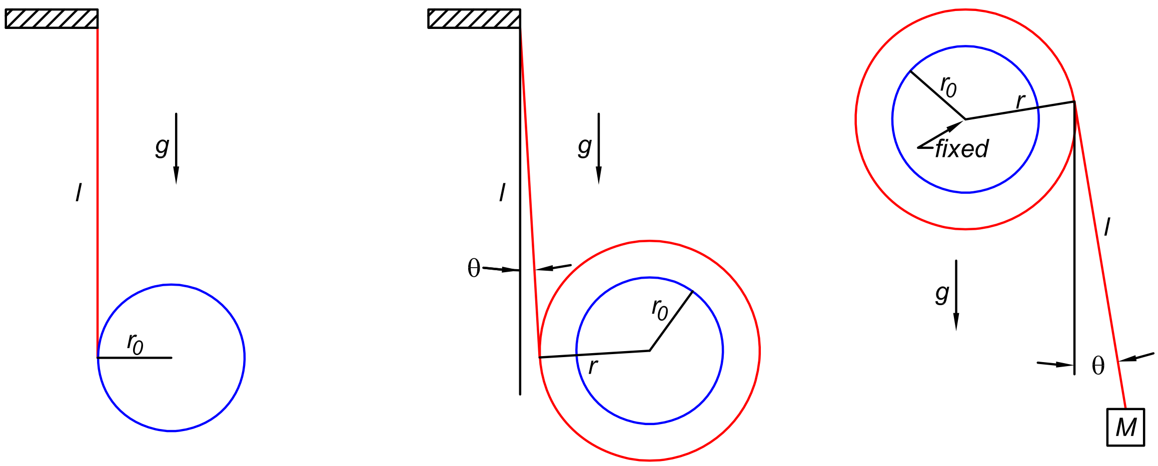
1 Problem

In the classic yo-yo problem a spool of mass m, radius r_0, and moment of inertia $I = kmr_0^2$ about it axis has a massless, infinitely thin string wrapped around the radius r_0, with one end of the string fixed to a support above the spool, as shown in the left figure below.



Consider also the case that the string is a tape of length L, thickness t and mass per unit length ρ, with either the end of the tape fixed as in the middle figure above, or with the axis of the spool fixed and mass M attached to the end of the tape.

Discuss the motion in all three cases, assuming that the length l of the unwound portion of the string/tape is initially zero.

2 Solution

2.1 Massless String

The vertical speed v of the center of mass of the spool is related to its angular velocity ω by $v = \dot{l} = \omega r_0$. Hence, the (constant) energy of the yoyo, which starts from rest with $l = 0$ at time $t = 0$, is

$$E = 0 = \frac{mv^2}{2} + \frac{I\omega^2}{2} - mgl = \frac{m(1+k)\dot{l}^2}{2} - mgl. \tag{1}$$

Thus,

$$\dot{l}^2 = \frac{2gl}{1+k}, \qquad \ddot{l} = \frac{g}{1+k}, \qquad l = \frac{gt^2}{2(1+k)}. \tag{2}$$

For the case of a solid cylinder of radius r_0, $k = 1/2$, $\ddot{l} = 2g/3$ and $l = gt^2/3$.

2.2 Yo-Yo with Massive Tape

We next consider a yo-yo suspended by a tape, rather than a string, of length L, thickness t, linear mass density ρ and mass $m_t = \rho L$. When length l of the tape has been unwound, the remaining tape on the yo-yo has area $(L - l)t = \pi(r^2 - r_0^2)$, where

$$r = \sqrt{\frac{\pi r_0^2 + (L - l)t}{\pi}} \tag{3}$$

is the outer radius of the tape on the yo-yo. Then, the mass and moment of inertia of the yo-yo are

$$m_y = m + \rho(L - l), \qquad I_y = k m r_0^2 + \frac{\pi\rho(r^4 - r_0^4)}{2t}\,. \tag{4}$$

In the approximation that the length l of the unwound tape is straight, the system has only two degrees of freedom, which we take to be the length l and the angle θ of that length to the vertical. Then, taking the origin to be at the support point of the tape, the center of the yo-yo is at position

$$x_y = l\sin\theta + r\cos\theta, \qquad y_y = -l\cos\theta + r\sin\theta, \tag{5}$$

which point has velocity

$$\dot{x}_y = \dot{l}\sin\theta + l\cos\theta\,\dot{\theta} + \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}, \qquad \dot{y}_y = -\dot{l}\cos\theta + l\sin\theta\,\dot{\theta} + \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}. \tag{6}$$

The velocity of the point on the tape that is just about to lose contact with the winding, relative to the center of the yo-yo is $\dot{l} - r\dot{\theta}$, so the angular velocity of the yo-yo is

$$\omega = \frac{\dot{l} - r\dot{\theta}}{r} = \frac{\dot{l}}{r} - \dot{\theta}. \tag{7}$$

The kinetic energy of the system is

$$T = \frac{\rho l^3\dot{\theta}^2}{6} + \frac{m_y}{2}\left(\dot{l}^2 + \dot{r}^2 + (l^2 + r^2)\,\dot{\theta}^2 + 2l\dot{r}\,\dot{\theta} - 2r\dot{l}\,\dot{\theta}\right) + \frac{I_y\omega^2}{2}\,, \tag{8}$$

and the potential energy is

$$V = -\frac{\rho g l^2\cos\theta}{2} + m_y g y_y. \tag{9}$$

Lagrange's method can now be used to deduce the equations of motion for coordinates l and θ, but these are somewhat complicated. Here, we content ourselves with yet another approximation, that angle θ is negligibly small. In this case, the kinetic energy of the system is

$$T \approx \frac{m_y}{2}\left(\dot{l}^2 + \dot{r}^2\right) + \frac{I_y\dot{l}^2}{2r^2} \qquad (\theta = 0), \tag{10}$$

and the potential energy is

$$V = -\frac{\rho g l^2}{2} - m_y g l \qquad (\theta = 0). \tag{11}$$

This is still somewhat complicated, so we restrict our attention to the case of a roll of tape wound on itself, *i.e.*, $m = 0 = r_0$, for which

$$r = \sqrt{\frac{(L-l)t}{\pi}}\,, \qquad \dot{r} = -\frac{t\dot{l}}{2\sqrt{\pi(L-l)t}}\,, \qquad m_y = \rho(L-l), \qquad I_y = \frac{\pi\rho r^4}{2t}\,, \tag{12}$$

$$T \approx \frac{3\rho\dot{l}^2(L-l)}{4} + \frac{\rho\dot{l}^2 t}{8\pi} \qquad (\theta = m = r_0 = 0), \tag{13}$$

and the potential energy is

$$V = -\rho g l\left(L - \frac{l}{2}\right) \qquad (\theta = m = r_0 = 0). \tag{14}$$

The total energy is zero, so $T = -V$, and

$$\dot{l}^2\left(1 + \frac{t}{6\pi(L-l)}\right) = \frac{2gl}{3}\frac{2L-l}{L-l} \qquad (\theta = m = r_0 = 0). \tag{15}$$

Until the tape is almost completely unwound, $6\pi(L-l) \gg t$, during which time

$$\dot{l}^2 \approx \frac{2gl}{3}\frac{2L-l}{L-l}\,, \qquad \ddot{l} \approx g\frac{2L(L-l)+l^2}{3(L-l)^2} \qquad (\theta = m = r_0 = 0). \tag{16}$$

The initial acceleration (when $l \approx 0$) is again $2g/3$, as found in sec. 2.1. The acceleration of the center of the roll of tape grows as it unwinds, reaching a maximum when $L - l \approx t$ with $\ddot{l}_{\max} \approx gL^2/3t^2 \gg g$.

Another approximation is that the tape is wound on a massless spool of radius r_0 and that the thickness t of the tape is negligible compared to r_0. In this case the kinetic energy is $T = \rho(L-l)\dot{l}^2$ and the potential energy is $V = -\rho g l(L - l/2)$, such that

$$\dot{l}^2 = g\frac{l(L-l/2)}{L-l}\,, \qquad \ddot{l} = g\frac{2L(L-l)+l^2}{4(L-l)^2}\,. \tag{17}$$

Both $\dot{l}$ and $\ddot{l}$ diverge as l approaches L.

As discussed in [1, 2, 3, 4], if a roll of tape unwinds down an inclined plane, the end of the unrolling tape strikes the plane with very high speed, making a loud sound.

2.3 Unwinding Spool with Fixed Axle

For the third example sketched on p. 1, where the spool has a fixed axle and mass M hangs from the end of the tape, eqs. (3)-(7) hold again, but with the interpretation that (x_y, y_y) is the position of mass M. Then, $(\dot{x}_y, \dot{y}_y)$ is the velocity of mass M, and of the lower end of the unwound tape. The velocity of the upper end of the unwound tape is the sum of the velocity of the point $\mathbf{p} = (r\cos\theta, r\sin\theta)$ where the tape leaves the spool and the velocity $\boldsymbol{\omega} \times \mathbf{p} = \omega(-p_y, p_x)$ of the tape relative to that point. That is,

$$\dot{x}_{\rm upper} = \dot{p}_x - \omega p_y, \qquad \dot{y}_{\rm upper} = \dot{p}_y + \omega p_x, \tag{18}$$

such that the velocity of the center of mass of the unwound tape is

$$\dot{x}_u = \frac{\dot{x}_y + \dot{x}_{\rm upper}}{2}, \qquad \dot{y}_u = \frac{\dot{x}_y + \dot{x}_{\rm upper}}{2}. \tag{19}$$

The kinetic energy of this system is

$$T = \frac{M}{2}(\dot{x}_y^2 + \dot{y}_y^2) + \frac{\rho l}{2}(\dot{x}_u^2 + \dot{y}_u^2) + \frac{\rho l^3\, \dot{\theta}^2}{24} + \frac{I_y \omega^2}{2}, \tag{20}$$

Recalling that ρ is the linear mass density of the tape, the potential energy is

$$V = Mgy_y + \rho l g y_u. \tag{21}$$

In the approximations that the angle θ of the tape to the vertical is negligible, and that terms in $\dot{r}$ are negligible, then $\dot{\mathbf{p}} = 0$, $\omega = \dot{l}/r$, $\dot{x}_u = \dot{x}_y = 0$, $\dot{y}_u = \dot{y}_y = -\dot{l}$, and the kinetic energy becomes

$$\begin{aligned} T &\approx \frac{\dot{l}^2}{2}\left(M + \rho l + \frac{I_y}{r^2}\right) \\ &= \frac{\dot{l}^2}{2}\left(M + \rho l + \frac{\pi k m r_0^2}{\pi r_0^2 + (L-l)t} + \frac{\rho[\pi r_0^2 + (L-l)t]}{2t} - \frac{\pi^2 \rho r_0^4}{2t[\pi r_0^2 + (L-l)t]}\right), \end{aligned} \tag{22}$$

while the potential energy is now

$$V \approx -Mgl - \frac{\rho l^2 g}{2}. \tag{23}$$

The total energy is zero, so again $T = -V$, and

$$\dot{l}^2\left(M + \rho l + \frac{\pi k m r_0^2}{\pi r_0^2 + (L-l)t} + \frac{\rho[\pi r_0^2 + (L-l)t]}{2t} - \frac{\pi^2 \rho r_0^4}{2t[\pi r_0^2 + (L-l)t]}\right) = 2Mgl + \rho l^2 g. \tag{24}$$

The resulting expression for $\dot{l}$ could be integrated numerically to find $l(t)$.

A somewhat trivial case is that the hanging mass M is large compared to the mass of the spool and tape, for which $\ddot{l} = g$. Another special case is that the spool has zero radius, $r_0 = 0$, such that

$$\dot{l}^2\,[2M + \rho(L+l)] = 2gl(2M + \rho l), \qquad \ddot{l} = g\frac{2M[2M + \rho(L+2l)] + \rho^2 l(2L+l)}{[2M + \rho(L+l)]^2}. \tag{25}$$

If there is no hanging mass M, the acceleration of the unwound tape becomes

$$\dot{l}^2 = g\frac{2l^2}{L+l}\,, \qquad \ddot{l} = g\frac{l(2L+l)}{(L+l)^2}\,, \tag{26}$$

which starts out at zero and rises to $3g/4$ when the tape is fully unwound.[1] The center of mass of the tape is at

$$y_{\rm cm} = -\frac{l^2}{2L}\,, \qquad \ddot{y}_{\rm cm} = -\frac{l\,\ddot{l} + \dot{l}^2}{L} = -g\frac{l^2(4L+3l)}{L(L+l)^2}\,. \tag{27}$$

As the tape becomes fully unwound, the acceleration of the center of mass of the tape is $\ddot{y}_{\rm cm} \to -7g/4$, but once the tape is fully unwound $\ddot{y}_{\rm cm}$ must be $-g$.[2]

Acknowledgment

Thanks to Johann Otto for discussions of this problem.

References

[1] L.T. Pockman, *Nonconservation of Energy—A Paradox*, Am. J. Phys. **9**, 50 (1941), http://physics.princeton.edu/~mcdonald/examples/mechanics/pockman_ajp_9_50_41.pdf

[2] I.A. Freeman, *The Dynamics of a Roll of Tape*, Am. J. Phys. **14**, 124 (1946), http://physics.princeton.edu/~mcdonald/examples/mechanics/freeman_ajp_14_124_46.pdf

[3] L.T. Pockman, *More on the Dynamics of a Roll of Tape*, Am. J. Phys. **14**, 275 (1946), http://physics.princeton.edu/~mcdonald/examples/mechanics/pockman_ajp_14_275_46.pdf

[4] P. Calvini, *Dynamics of a tape that unwinds while rolling down an incline*, Am. J. Phys. **51**, 226 (1983), http://physics.princeton.edu/~mcdonald/examples/mechanics/calvini_ajp_51_226_83.pdf

[1] The angular velocity of the still-wound tape diverges as the unwound length l approaches L, and the approximation that the unwound tape is entirely vertical breaks down.

[2] This discrepancy is further evidence that the approximations used here do not hold well as the length l of the unwound tape approaches L, for the idealized case that $M = 0$ and $r_0 = 0$.