

A Bounded Source Cannot Emit a Unipolar Electromagnetic Wave

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1 Problem

Show that a bounded source cannot produce a unipolar electromagnetic pulse.

Equivalently, show that there are no three-dimensional electromagnetic solitons in vacuum.

2 Solution

This problem was first discussed by Stokes over 150 years ago [1, 2, 3, 4], who noted that three-dimensional sound waves from a bounded source cannot be unipolar. While his argument applies to electromagnetic waves as well, this is little recognized in the literature. One-dimensional electromagnetic (and sound) waves can be unipolar.¹ A plane wave can have any pulseform, but, strictly speaking, a plane wave can only be generated by an unbounded source. A well-known pedagogic example given in sec. II-20 of [6] is based on this case.

Hence, this old problem still has a new flavor. Here, we offer two new solutions, followed by discussion.

2.1 Via Conservation of Energy and Fourier Analysis

If the source is bounded, it appears pointlike when viewed from a great enough distance. Then, energy conservation requires that the pulse energy density fall off as $1/r^2$, for distance r measured from some characteristic point within the source. Since the energy density is proportional to the square of the electromagnetic fields, we have the well-known result that the radiation fields from a bounded source fall off as $1/r$ far from the source.

This is in contrast to the static fields, which must fall off at least as quickly at $1/r^2$, far from a bounded source.

¹Here, a *one dimensional* wave means a function in a space of three spatial dimensions (plus time, *i.e.*, $3 + 1$ spacetime) that depends on only one spatial coordinate. In contrast, electrodynamics in $1 + 1$ spacetime is rather trivial, with no magnetic field and no electromagnetic waves, as reviewed, for example, in [5].

Now, consider the possibility of a unipolar pulse, *i.e.*, one for which the electric field components $E_i(\mathbf{r}, t)$ have only one sign. At a fixed position \mathbf{r} , the time integral of at least one component of such a pulse would be nonzero,

$$\int E_i(\mathbf{r}, t) dt \neq 0. \quad (1)$$

Then a Fourier analysis of this component,

$$E_i(\mathbf{r}, \omega) = \int E_i(\mathbf{r}, t) e^{i\omega t} dt, \quad (2)$$

would have a nonzero value at zero frequency, $E_i(\mathbf{r}, \omega = 0) \neq 0$.

However, the quantity $E_i(\mathbf{r}, \omega = 0)$ would then be a static solution to Maxwell's equation, and so must fall off like $1/r^2$. This contradicts the hypothesis that $E_i(\mathbf{r}, t)$ represented an electromagnetic pulse from a bounded source, which must fall off as $1/r$.

Thus, a bounded source cannot emit a unipolar electromagnetic pulse.

2.2 Via the Fields of an Accelerated Charge

The electric field vector radiated by a charge is opposite to the transverse component of the acceleration.

In the case of a bounded source, the accelerations of the charges cannot impart a nonzero average velocity to any charge; otherwise the source would take on a nonzero recoil velocity and not remain bounded.

Hence, any accelerations must include both positive and negative components such that their time integral vanishes.

Therefore, the radiated electric field must also include both positive and negative components.

Again, A bounded source cannot emit a unipolar electromagnetic pulse.

A variant of the above argument using the Liénard-Wiechert fields has been given by Bessonov [7], who also considers the case of a bipolar pulse consisting of a pair of well separated, opposite-sign unipolar pulses.

2.3 Three-Dimensional Unipolar Radiation from an Unbounded Source

In the case of two nonrelativistic, unbound charged particles that interact only via the Coulomb force $q_1 q_2 / r$, the component of the acceleration of one of the charges along the axis of its hyperbolic trajectory always has the same sign. Hence, the radiated electric field component along that axis is unipolar, and a Fourier analysis of the field has a zero-frequency component (see sec. 70 of [8]). Thus, a three-dimensional unipolar pulse can be emitted by a system whose motion is unbounded.

2.4 Unipolarlike Pulses

It is possible for a bounded system to produce an electromagnetic pulse that consists almost entirely of a single central pulse of one sign. But according to the argument above, this pulse must include long tails of the opposite sign so that the time integral of the fields vanish at any point far from the source. This behavior has been observed in several recent reports on subcycle electromagnetic pulses [9, 10, 11, 12].

3 Can the Far-Zone Radiation From a Bounded Source Transfer Net Energy to a Charged Particle?

The present considerations can be extended to comment on this topical question.

If a unipolar electromagnetic pulse existed far from its bounded source, the corresponding vector potential \mathbf{A} would have different values at asymptotically early and late times. Then, as argued by Lai [13], an interaction with a charge e and mechanical momentum \mathbf{P} that conserves the canonical momentum $\mathbf{P} + e\mathbf{A}/c$ could produce a net change in the magnitude of the mechanical momentum, *i.e.*, provided a transfer of energy between the particle and the pulse. Such far-zone unipolar particle acceleration is desirable, but is not consistent with Maxwell's equations.

Near-zone particle acceleration by time-dependent electromagnetic fields can be accomplished by passing a particle through a bounded field region, such as an rf cavity, during a short interval when the fields have only one sign. While the electromagnetic fields are not unipolar in this case, their interaction with the charged particle is effectively so.

A variant on such considerations is the fact that a bounded electrostatic field cannot exchange net energy with a charged particle that begins and ends its history at large distances from the source. So-called electrostatic particle accelerators all contain a nonelectrostatic component that can move the charge to a region of nonzero electric potential and leave it there with effectively zero electrical and mechanical energy. Then, the charge can extract energy from the field as it is expelled to large distances.

Returning to the case of far-zone radiation, we give a brief argument based the well-known relation for the time rate of change of energy U of a particle of charge e and velocity \mathbf{v} in an electromagnetic field \mathbf{E} (see, for example, eq. (17.7) of [8]),

$$\frac{dU}{dt} = e\mathbf{E} \cdot \mathbf{v} \quad (3)$$

This expression holds for particles of any velocity less than the speed of light. Of course, the magnetic field cannot change the particle's energy.

In the approximation that the particle's velocity is essentially unchanged by its interaction with the electromagnetic field, we have,

$$\Delta U = e\mathbf{v} \cdot \int \mathbf{E} dt. \quad (4)$$

To perform the integral, we can use Feynman's expression for the far-zone radiated electric

field of an accelerated charge (sec. I-34 of [6]),

$$\mathbf{E}_{\text{rad}} = \frac{e}{c^2} \frac{d^2 \hat{\mathbf{n}}}{dt^2}, \quad (5)$$

in Gaussian units, where $\hat{\mathbf{n}}$ is a unit vector from the retarded position of the source charge to the observer. Then, $\int \mathbf{E}_{\text{rad}} dt$ is the difference between $d\hat{\mathbf{n}}/dt$ at early and late times. Since $d\hat{\mathbf{n}}/dt$ is the angular velocity of the relative motion between the source and charge, this vanishes at both early and late times as the moving charge is then arbitrarily far from the bounded source. Hence, in the constant-velocity approximation, the far-zone fields from a bounded source cannot transfer energy to a particle.

While the constant-velocity approximation is not necessarily good for a particle whose initial velocity is nonrelativistic, it is an excellent approximation for a relativistic particle. It is possible for the far-zone fields of a bounded source (in vacuum and far from any other material) to transfer energy to a nonrelativistic charge, as recently observed [14], but this energy transfer becomes increasingly inefficient as the particle's velocity approaches that of light.

The above considerations have ignored energy transfers due to scattering, which can be significant if the energy of a photon of the electromagnetic wave is large compared to the total energy of the charged particle in the center of mass frame. In this case, a quantum description is more appropriate. In the regime in which the energy transfer from a single scattering process is small, the classical idea of a “radiation pressure” associated with the Poynting vector $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{H}$ can be formalized by including the radiation reaction in the equation of motion of the charged particle. See, for example, eq. (75.10) of [8]. However, this effect is always very small in the classical regime.

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