Buquoy's Problem: Lifting a String from a Table

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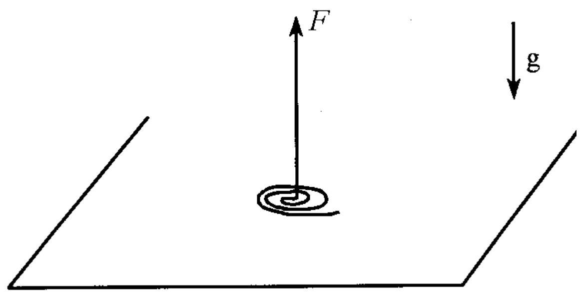
1 Problem

What force $F$is required to lift an inextensible string at constant acceleration a from a table, if effects of motion of the string on the table can be ignored.



Discuss also the cases of a constant upward force F, and of free fall of the string onto the table.

This example was first considered by Buquoy (1815) [1] (which may have been the first analysis of a variable-mass problem; see also [2]).

2 Solution

For a string of mass λ per unit length whose upper end is at height h above the table, the total (vertical) force on the portion of the string above the table is $F - \lambda gh$, where F is the force on the upper end of the string. In this, we have ignored any possible force of the table on the string that has been lifted. The momentum of the string is $p = \lambda h\dot{h}$, such that

$$F_{\rm tot} = F - \lambda gh = \dot{p} = \lambda h\ddot{h} + \lambda\dot{h}^2, \qquad F = \lambda h(g + \ddot{h}) + \lambda\dot{h}^2. \tag{1}$$

For constant acceleration a beginning at $t = 0$ when $h = h_0$ and $\dot{h} = v_0$, we have for $t > 0$ that

$$\dot{h} = v_0 + at, \qquad h = h_0 + v_0 t + \frac{at^2}{2}, \qquad F = \lambda\left[\left(h_0 + v_0 t + \frac{at^2}{2}\right)(a + g) + (v_0 + at)^2\right]. \tag{2}$$

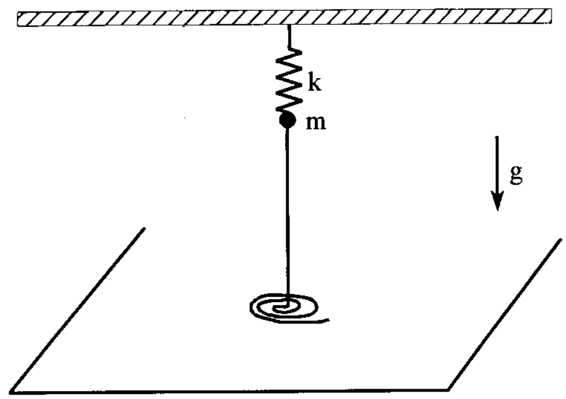
A nonzero initial velocity v_0 implies that a large impulse was applied at $t = 0$; in this scenario the force required to lift the string at this constant velocity for $t > 0$ would be $F = \lambda[g(h_0 + v_0 t) + v_0^2]$.

A delicacy in the above argument concerns the tension T in the string just above the table. This must be nonzero, since this is the force that accelerates length $\dot{h}\,dt$ of the string from rest on the table to vertical velocity $\dot{h}$ during time dt. The change in momentum during time dt of this segment of the string is $dp = \lambda\dot{h}^2\,dt$, which implies that $T = \lambda\dot{h}^2$.

If we suppose that there exists an equal and opposite force on the string, due to the string on the table, then in eq. (1) the total force on the portion of the string off the table would be $F - \lambda g h - T$, which would imply that $F = \lambda h(g + \ddot{h}) + 2\lambda \dot{h}^2$. However, this would be "double counting" of the term $\lambda \dot{h}^2$, and eqs. (1)-(2) are correct as is.

Moral: it is delicate to use Newton's third law in variable-mass problems.[1]

An elaboration of the present example is discussed in sec. 4 of [3], in which the upper end of the string is attached to a mass that is suspended from a ceiling by a spring.



2.1 Constant Upward Force

The equation of motion follows from eq. (1) as

$$\frac{d\,h\dot{h}}{dt} = \frac{F}{\lambda} - gh, \qquad h\dot{h}\frac{d\,h\dot{h}}{dt} = \frac{F}{\lambda}h\dot{h} - gh^2\dot{h}, \tag{3}$$

which integrates to

$$\frac{1}{2}(h\dot{h})^2 = \frac{F}{\lambda}\frac{h^2}{2} - \frac{gh^3}{3}, \qquad \dot{h}^2 = \frac{F}{\lambda} - \frac{2gh}{3}, \tag{4}$$

supposing that $h = 0 = \dot{h}$ at time $t = 0$. Note that if F is small enough, the increasing weight of the string off the table will reduce the upward velocity to zero when $h = 3F/2g\lambda$. If this h is less than the length l of the string, *i.e.*, $F < 2g\lambda/3$, the string falls back onto the table (as will be confirmed after eq. (6) below).

Supposing that F is larger than this, eq. (4) leads to

$$t = \int_0^h \frac{dh}{\sqrt{F/\lambda - 2gh/3}} = \frac{3}{g}\left(\sqrt{\frac{F}{\lambda}} - \sqrt{\frac{F}{\lambda} - \frac{2gh}{3}}\right), \qquad h = \sqrt{\frac{F}{\lambda}}t - \frac{gt^2}{6}, \tag{5}$$

up to time

$$t = \frac{3}{g}\sqrt{\frac{F}{\lambda}}\left(1 - \sqrt{1 - \frac{2g\lambda l}{3F}}\right), \tag{6}$$

[1]This point has been emphasized in [4, 5], and is illustrated in the author's examples [6, 7].

when the string is lifted completely off the table.

However, if $F < 2g\lambda/3$, the string reaches a maximum height $h_{\rm max} = 3F/2g\lambda$ at time $t = 3\sqrt{F/\lambda}/g$, after which it falls freely back onto the table. The time of the free fall is $\Delta t = \sqrt{2h_{\rm max}/g} = \sqrt{3F/\lambda}/g$, and the motion of the strings lasts only for a total time $(3+\sqrt{3})\sqrt{F/\lambda}/g$.

2.2 Free Fall of the String onto the Table

If the lower end of a freely falling string reaches the table with velocity v, the potion of the string not yet on the table continues to fall freely for additional time $(v/g)(\sqrt{1 + 2gl/v^2} - 1)$, after which the string is at rest on the table. For $v = 0$, the additional fall time is $\sqrt{2l/g}$, while for large v it is just l/v.

References

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