Single-Bubble Sonoluminescence

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1 Problem

In the phenomenon of single-bubble sonoluminescence, a water bubble of initial radius of 40 μm is observed to emit light when its radius collapses to about 0.5 μm under one atmosphere pressure. Approximately 6×10^6 photons are emitted in the energy range 1-6 eV, with a bremsstrahlung-like spectrum of the form $dN \propto dE/E$ where E is the photon energy. (Water is opaque to photons above about 6 eV.)

In this problem the bubble can assumed to contain vacuum.

a) Suppose that all the kinetic energy of the collapsing bubble is converted to photons with the spectrum $dN \propto dE/E$. What would be the maximum photon energy emitted?

b) At what radius does the velocity of the inner surface of the bubble reach the speed of sound in water, 1,500 m/s?

2 Solution

This problem is based on measurements by S. Putterman *et al.*, Phys. Rev. Lett. **79**, 1799 (1997), and references therein.

a) The bubble is a spherical cavity of initial radius R_0 in an infinite reservoir of (incompressible) water at pressure $P = 1$ atm $= 10^5$ N/m^2 far from the bubble. As the bubble collapses, the pressure does work on the decreasing volume, which energy goes to increasing the kinetic energy of the moving fluid (Rayleigh, Phil. Mag. **34**, 94 (1917)).

When the bubble has collapsed to radius R its volume has changed by amount

$$\Delta V = \frac{4\pi}{3}(R_0^3 - R^3), \tag{1}$$

so the work done is

$$KE = W = P\Delta V = \frac{4\pi P}{3}(R_0^3 - R^3) \approx \frac{4\pi R_0^3 P}{3}\,, \tag{2}$$

where the approximation holds once $R \ll R_0$.

For example, if a 40-μm-radius water bubble collapses under 1 atmosphere pressure, the maximum kinetic energy of the system is (in MKSA units)

$$KE_{\text{max}} \approx \frac{4\pi(4 \times 10^{-5})^3 10^5}{3} \approx 2.5 \times 10^{-8} \text{ J} \approx 1.5 \times 10^{11} \text{ eV}. \tag{3}$$

Suppose that all the energy is dissipated in sonoluminescence with a photon spectrum $dN \propto d\nu/\nu$, and that at frequencies above 1 eV but below the water transmission cutoff (6 eV) a total of 6×10^6 photons are observed.

We write the photon spectrum as $dN = a dE/E$. Then the observed number of photons determines parameter a via

$$6 \times 10^6 = a \int_1^6 \frac{dE}{E} = a \ln 6 = 1.8a, \tag{4}$$

so $a = 3.3 \times 10^6$.

The maximum photon energy, $E_{\rm max}$, is now determined by

$$KE = 1.5 \times 10^{11} \text{ eV} = \int_0^{E_{\rm max}} E\ dN = a \int_0^{E_{\rm max}} dE = aE_{\rm max} = 3.3 \times 10^6 E_{\rm max}. \tag{5}$$

Thus $E_{\rm max} = 45{,}000$ eV = 45 keV.

b) If the surface of the bubble has (radial) velocity $v(R)$, then the fluid velocity at a radius $r > R$ is also radial, with a value that follows from the continuity equation

$$0 = \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{dr^2 v}{dr}\ . \tag{6}$$

[Use Gauss' law if don't remember the divergence in spherical coords.] Hence, $r^2 v$ is constant throughout the fluid at any given time, and so

$$v(r) = \frac{R^2}{r^2} v(R). \tag{7}$$

The total kinetic energy of the moving fluid, of density ρ, is

$$KE = \int_R^\infty 4\pi r^2\ dr \frac{\rho v^2(r)}{2} = 2\pi\rho v^2(R) R^4 \int_R^\infty \frac{dr}{r^2} = 2\pi\rho v^2(R) R^3 \tag{8}$$

Equating (2) and (8) we find that

$$v^2(R) = \frac{2P}{3\rho}\left(\frac{R_0^3}{R^3} - 1\right) \approx \frac{2P}{3\rho}\frac{R_0^3}{R^3}\,, \tag{9}$$

where the approximation holds once $R \ll R_0$

The inner surface of a water bubble of initial radius $R_0 = 40\ \mu$m reaches velocity $v_{\rm sound} \approx 1{,}500$ m/s at radius

$$R = R_0 \sqrt[3]{\frac{2P}{3\rho v_{\rm sound}^2}} = 40\ \mu\text{m}\ \sqrt[3]{\frac{2 \times 10^5}{3 \cdot 10^3 \cdot (1.5 \times 10^3)^2}} = 1.4\ \mu\text{m}. \tag{10}$$

This radius is about three times larger than that at which the sonoluminescence is observed to occur. Since the water velocity is hypersonic for $R < 1.4\ \mu$m it is likely that other dissipative mechanisms besides sonoluminescence are involved, and that maximum photon energy is less than 45 keV as found in part a).