Slingshot Ride

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1 Problem

A popular ride at amusement parks is the "slingshot," in which two bungee cords of rest length l_0 and spring constant k are attached between two poles distance $2l$ apart and connected to mass m. The mass is lowered by height $H > 0$ below the tops of the poles, and then released.
What is the maximum velocity of the mass?
What is the maximum height h above the tops of the poles reached by the mass? For this, suppose that $l_0 = 0$.
What are the frequencies of the normal modes of small oscillation of the system about equilibrium?

2 Solution

We assume that there is no energy dissipation in the bungee cords. Then for purely vertical motion along the z-axis, with $z = 0$ at the top of the poles, the energy is

$$\begin{aligned} E &= \frac{mv^2}{2} + k\left(\sqrt{z^2 + l^2} - l_0\right)^2 + mgz \\ &= k\left(\sqrt{H^2 + l^2} - l_0\right)^2 - mgH \\ &= k\left(\sqrt{h^2 + l^2} - l_0\right)^2 + mgh \\ &= \frac{mv_{\rm max}^2}{2}\,. \end{aligned} \tag{1}$$

The maximum velocity occurs when the mass passes by $z = 0$, where

$$v_{\rm max} = \sqrt{\frac{2k}{m}\left[H^2 + 2l_0\left(\sqrt{H^2 + l^2} - l\right)\right] - 2gH} \quad \to \quad \sqrt{\frac{2kH^2}{m} - 2gH} \quad \text{if } l_0 = 0. \tag{2}$$

To find the maximum height h we equate the second and third lines of eq. (1), which leads to a quartic equation in h if $l_0 > 0$. To obtain a simple analytic result we suppose that $l_0 = 0$, in which case we find only a quadratic equation in h,

$$h^2 + \frac{mgh}{k} + \frac{mgH}{k} - H^2 = 0 = (h + H)\left(h - H + \frac{mg}{k}\right), \tag{3}$$

so that the maximum height is

$$h = H - \frac{mg}{k}\,. \tag{4}$$

The general motion is in all three coordinates x, y and z, where we take the x-axis along the line connecting the tops of the poles. One normal mode involves purely vertical oscillations, and another is simple pendulum motion in the y-z plane. The third normal mode is orthogonal to the first two, so should involve oscillation only in x.

For purely vertical motion,

$$m\ddot{z} = -mg - \frac{2kz}{\sqrt{z^2+l^2}}\left(\sqrt{z^2+l^2} - l_0\right). \tag{5}$$

Again, an analytic description is much simpler if $l_0 = 0$. Then,

$$m\ddot{z} = -mg - 2kz, \tag{6}$$

for which the equilibrium is at

$$z_0 = -\frac{mg}{2k}, \tag{7}$$

and the angular frequency of small oscillations is

$$\omega_1 = \sqrt{\frac{2k}{m}}. \tag{8}$$

The second mode is simple pendulum motion in the y-z plane with length $|z_0| = mg/2k$. The angular frequency of small oscillations for this mode is

$$\omega_2 = \sqrt{\frac{g}{|z_0|}} = \sqrt{\frac{2k}{m}} = \omega_1. \tag{9}$$

The third mode is for oscillations along the horizontal line with $y = 0$, $z = z_0$, for which the equation of motion is

$$m\ddot{x} = -k\left(\frac{x}{\sqrt{(x-l)^2+z_0^2}}\sqrt{(x-l)^2+z_0^2} + \frac{x}{\sqrt{(2l-x)^2+z_0^2}}\sqrt{(2l-x)^2+z_0^2}\right) = -2kx. \tag{10}$$

The angular frequency of small oscillations for this mode is

$$\omega_2 = \sqrt{\frac{2k}{m}} = \omega_1 = \omega_2. \tag{11}$$

All three modes have the same frequency when $l_0 = 0$, and the system is equivalent to mass m being tied to the equilibrium point $(0, 0, z_0)$ by a spring of zero length and constant $2k$.