Greek Temple Seismograph

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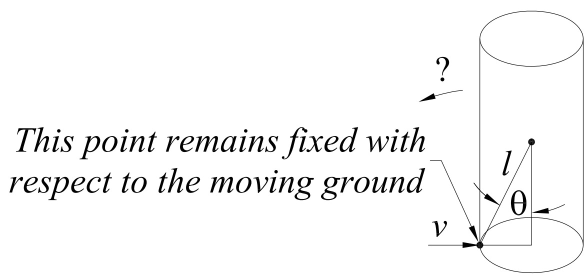
1 Problem

Following a major earthquake near Naples in 1857, R. Mallet suggested that a measure of the horizontal velocity of the ground during an earthquake could be deduced from the maximum height of cylindrical columns that remained standing [1].

Deduce the minimum horizontal velocity v needed to overturn a solid, vertical, cylindrical column whose diagonal has length $2l$ and makes angle θ to the vertical, assuming that a point on the base remains fixed with respect to the moving ground, as sketched below.



If the cylinder is too squat (large θ), it can lose contact with the ground during its motion. Supposing that the velocity v is the minimum value found above, deduce a condition on the angle θ such that the cylinder always remains in contact with the ground as it falls over.

2 Solution

This problem is taken from secs. 174-175 of [2].

The effect on the vertical cylinder of a sudden horizontal velocity v of the ground is equivalent to the case of the ground remaining at rest and the center of mass of the cylinder being given a sudden horizontal velocity v. In either view, a large impulsive force acts at the point on the base of the cylinder that remains fixed relative to the ground. This force exerts no torque about the fixed point, so the change ΔL in angular momentum of the cylinder during the impulse equals that associated with the sudden horizontal velocity of the center of mass (in the frame where the ground remains at rest). Hence,

$$\Delta L = mvl\cos\theta = I_p\,\omega_0, \qquad \text{and so} \qquad \omega_0 = \frac{mvl\cos\theta}{I_p}\,, \tag{1}$$

where m is the mass of the cylinder, I_p is its moment of inertial about the fixed point, and ω_0 is the initial angular velocity just after the impulse. Recalling that the moment of inertia of a thin disc about an a diameter is $mr^2/4$, the moment of inertia I_p is, using the parallel axis theorem,

$$I_p = \int_0^h \frac{m\ dy}{h}\left(\frac{r^2}{4} + r^2 + y^2\right) = \frac{m}{4}\left(5r^2 + \frac{4h^2}{3}\right) = \frac{ml^2}{12}(15\sin^2\theta + 16\cos^2\theta) = \frac{ml^2}{12}(15 + \cos^2\theta), \tag{2}$$

noting that $r = l\sin\theta$ and $h = 2l\cos\theta$.

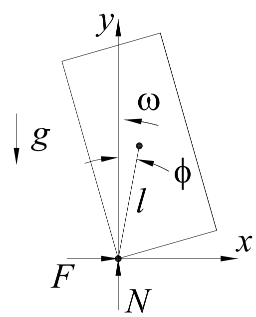
The column will fall over if the initial kinetic energy $I_p\omega_0^2/2$ just after the earthquake is sufficient that the center of mass of the column can rise from $h/2 = l\cos\theta$ to l. Hence, the minimum velocity of the ground needed to topple the column is related by

$$mgl(1-\cos\theta) = \frac{I_p\,\omega_{0,\min}^2}{2} = \frac{m^2 v_{\min}^2 l^2\cos^2\theta}{2I_p}\,, \tag{3}$$

and so,

$$v_{\min}^2 = \frac{2gI_p(1-\cos\theta)}{ml\cos^2\theta} = \frac{gl}{6}\frac{(1-\cos\theta)(15+\cos^2\theta)}{\cos^2\theta}\,. \tag{4}$$

As the column rotates about the fixed point with angular velocity $\omega(\phi)$, where ϕ is the angle of the diagonal to the vertical, it will lose contact with the ground if the normal force N goes to zero.



Referring to the figure above, the y-equation of motion of the center of mass of the cylinder is

$$F_y = m\ddot{y} = m\frac{d^2}{dt^2}(l\cos\phi) = ml\frac{d}{dt}(\omega\sin\phi) = ml(\dot{\omega}\sin\phi - \omega^2\cos\phi) = N - mg, \tag{5}$$

noting that $d\phi/dt \equiv \dot{\phi} = -\omega$. The normal force goes to zero if there is an angle ϕ such that

$$g = l(\omega^2\cos\phi - \dot{\omega}\sin\phi). \tag{6}$$

As the column rotates (about the z-axis), conservation of energy relates ω and ϕ according to

$$\omega^2 = \omega_0^2 - 2\frac{mgl}{I_p}(\cos\phi - \cos\theta). \tag{7}$$

If we restrict our attention to the case that the velocity of the ground is the minimum value (4), then using eq. (3) in (7) yields

$$\omega^2 = \frac{2mgl}{I_p}(1-\cos\phi). \tag{8}$$

Taking the time derivative of eq. (8) we find that

$$\dot{\omega} = -\frac{mgl}{I_p} \sin\phi, \tag{9}$$

and the condition (6) becomes

$$\frac{I_p}{mgl^2} = \frac{15 + \cos^2\theta}{12} = 2\cos\phi(1 - \cos\phi) + \sin^2\phi = -3\cos^2\phi + 2\cos\phi + 1, \tag{10}$$

or

$$3\cos^2\phi - 2\cos\phi + \frac{3 + \cos^2\theta}{12} = 0. \tag{11}$$

Thus, the column will loose contact with the ground if

$$\cos\phi = \frac{2 \pm \sqrt{4 - 12(3 + \cos^2\theta)/12}}{6} = \frac{2 \pm \sin\theta}{6}\,. \tag{12}$$

As the column rotates, angle ϕ is less than θ (and $\phi_0 = \theta$) so that $\cos\phi \geq \cos\theta$. If the column is to remain in contact with the ground at all times, we must have that neither solution (12) is greater than $\cos\theta$, *i.e.*,

$$\cos\theta > \frac{2 + \sin\theta}{6}\,. \tag{13}$$

The critical angle is roughly 61.3°, so if the height of the column is more than 1.83 times its diameter it will remain in contact with the ground at all times while it falls over after an earthquake that is minimally capable of causing this.

References

[1] R. Mallet, *The First Principles of Observational Seismology* (Chapman and Hall, London, 1862), vol. 1, chap. 16,
http://physics.princeton.edu/~mcdonald/examples/mechanics/mallet_chap16.pdf

[2] E.J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 7th ed. (Macmillan, London, 1905; reprinted by Dover Publications, 1960),
http://physics.princeton.edu/~mcdonald/examples/mechanics/routh_sec_174-175.pdf