

Electromagnetic Fields of a Rotating Shell of Charge

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1 Problem

A spherical shell of radius a carries charge Q uniformly distributed over its surface. Give expressions for the electric field \mathbf{E} and magnetic field \mathbf{B} everywhere in space when the shell is rotating with angular velocity ω about an axis. Compare these fields with those reported by an observer in a frame that rotates with angular velocity ω' about the same axis.

It suffices to consider the low-velocity case, where $\omega r \ll c$ and $\omega' r \ll c$ for any relevant radius r .

2 Solution

This problem will illustrate a remark by Schiff [1] that care must be taken when discussing electrodynamics in rotating frames, because the effect of the “distant stars” is to introduce “fictitious” charges and currents into the problem.

The example chosen here for a spherical shell of charge is perhaps slightly more complicated algebraically than the case of a cylindrical shell of charge, but the latter is less consistent with a universe in which “distant stars” play a role.

2.1 Rotating Spherical Shell in the Lab Frame

We work in a spherical coordinate system (r, θ, ϕ) with origin at the center of the sphere, and z axis along the axis of rotation, such that the angular velocity of the spherical shell is $\boldsymbol{\omega} = \omega \hat{\mathbf{z}}$. Then, in Gaussian units, the electric field is simply

$$\begin{aligned} \mathbf{E}(r < a) &= 0, \\ \mathbf{E}(r > a) &= \frac{Q}{r^2} \hat{\mathbf{r}}, \end{aligned} \tag{1}$$

independent of the angular velocity ω . If that angular velocity is nonzero, there is also a nonzero magnetic field in the lab frame.

A calculation of the magnetic field via the vector potential is given in sec. 5.4.1 of [2]. For possible contrast, we utilize a scalar potential Φ that can be defined everywhere except on the shell of radius a , since $\nabla \times \mathbf{B} = 0$ everywhere except on that surface. Thus, we write

$$\mathbf{B} = -\nabla\Phi, \tag{2}$$

where the potential Φ is not continuous across the surface $r = a$ because of the surface currents there. The potential is azimuthally symmetric, should be finite at the origin and

should vanish at infinity. Suitable forms for the regions inside and outside of the sphere are therefore

$$\begin{aligned}\Phi(r < a) &= \sum A_n \frac{r^n}{a^n} P_n(\cos \theta), \\ \Phi(r > a) &= \sum B_n \frac{a^{n+1}}{r^{n+1}} P_n(\cos \theta).\end{aligned}\quad (3)$$

We can obtain two matching condition across the surface $r = a$ from Maxwell's equations. First, $\nabla \cdot \mathbf{B} = 0$ implies that the radial component of \mathbf{B} , ie, $B_r = -\partial\Phi/\partial r$, is continuous at $r = a$. Hence,

$$\sum \frac{nA_n}{a} P_n(\cos \theta) = -\sum \frac{(n+1)B_n}{a} P_n(\cos \theta), \quad (4)$$

and therefore

$$A_n = -\frac{n+1}{n} B_n. \quad (5)$$

Also, Ampere's law, $\nabla \times \mathbf{B} = 4\pi\mathbf{J}/c$, applied to a small loop in a plane of constant ϕ that surrounds a segment of the charged surface, tells us that

$$\frac{4\pi}{c} K_\phi = B_\theta(r = a^+) - B_\theta(r = a^-) = -\frac{1}{a} \frac{\partial\Phi(r = a^+)}{\partial\theta} + \frac{1}{a} \frac{\partial\Phi(r = a^-)}{\partial\theta}, \quad (6)$$

where the surface current density \mathbf{K} due to the rotating charge is given by

$$\mathbf{K} = \frac{Q}{4\pi a^2} 2\pi a^2 \sin \theta \, d\theta \frac{\omega}{2\pi a} \frac{1}{d\theta} \hat{\phi} = \frac{\omega Q \sin \theta}{4\pi a} \hat{\phi} = -\frac{\omega Q}{4\pi a} \frac{d \cos \theta}{d\theta} \hat{\phi} = -\frac{\omega Q}{4\pi a} \frac{dP_1(\cos \theta)}{d\theta} \hat{\phi}. \quad (7)$$

Using eq. (7) in (6), we can integrate the latter to find,¹

$$\frac{\omega Q}{c} P_1 = \Phi(r = a^+) - \Phi(r = a^-) = \sum_n (B_n - A_n) P_n. \quad (8)$$

Since the P_n are a set of orthogonal functions, we see that $A_n = B_n = 0$ for $n \neq 1$, and that in view of eq. (5),

$$\frac{\omega Q}{c} = B_1 - A_1 = 3B_1. \quad (9)$$

Thus,

$$A_1 = -\frac{2\omega Q}{3c}, \quad B_1 = \frac{\omega Q}{3c}, \quad (10)$$

and

$$\begin{aligned}\Phi(r < a) &= -\frac{2\omega Q}{3ac} r P_1 = -\frac{2\omega Q z}{3ac}, \\ \Phi(r > a) &= \frac{\omega Q a^2 \cos \theta}{3c r^2}.\end{aligned}\quad (11)$$

It is useful to note that the magnetic moment \mathbf{m} of the spinning shell of charge is

$$\mathbf{m} = \frac{\hat{\mathbf{z}}}{c} \int_0^\pi \frac{Q}{4\pi a^2} 2\pi a^2 \sin \theta \, d\theta \frac{\omega}{2\pi} \pi (a \sin \theta)^2 = \frac{\omega Q a^2 \hat{\mathbf{z}}}{4c} \int_0^\pi \sin^3 \theta \, d\theta = \frac{\omega Q a^2 \hat{\mathbf{z}}}{3c} = \frac{Q a^2}{3c} \boldsymbol{\omega}. \quad (12)$$

¹Thanks to Marco Moriconi for pointing this out.

The potential can now be written

$$\begin{aligned}\Phi(r < a) &= -\frac{2m}{a^3}z, \\ \Phi(r > a) &= \frac{m \cos \theta}{r^2}.\end{aligned}\tag{13}$$

Thus, inside the shell the magnetic field is uniform, and outside the shell it has the form of a point dipole whose strength is the total magnetic dipole moment of the rotating shell,²

$$\begin{aligned}\mathbf{B}(r < a) &= \frac{2\mathbf{m}}{a^3}, \\ \mathbf{B}(r > a) &= \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}.\end{aligned}\tag{14}$$

2.2 Electrodynamics in a Rotating Frame

2.2.1 Summary of the Principles

We desire the fields \mathbf{E}' and \mathbf{B}' seen by an observer in a frame that rotates with angular velocity $\boldsymbol{\omega}'$ with respect to the lab frame, where $\boldsymbol{\omega}' = \omega' \hat{\mathbf{z}}$ is parallel to the angular velocity $\boldsymbol{\omega}$ of the spherical shell in the lab frame.³ We restrict our attention to points \mathbf{r}' in the rotating frame such that the velocity $\mathbf{v} = \boldsymbol{\omega}' \times \mathbf{r}'$ of the points with respect to the lab frame is small compared to the speed of light. Then, we can largely ignore issues of whether rods and clocks in the rotating frame measure the same lengths and time intervals as would similar rods and clocks in the lab frame, and whether the speed of light is the same in both frames. That is, we ignore Ehrerfest's paradox [7].

A discussion by the author of electrodynamics in a slowly rotating frame appears in the lengthy Appendix to [8].

For reference, we reproduce the principles of electrodynamics in the frame of a slowly rotating medium where ϵ and μ differ from unity.⁴

The (cylindrical) coordinate transformation is

$$r' = r, \quad \phi' = \phi - \omega' t, \quad z' = z, \quad t' = t,\tag{15}$$

²Another derivation is to note that the surface current (7) is the same as would hold for a uniformly magnetized sphere of radius a and magnetization $\mathbf{M} = Q\boldsymbol{\omega}/4\pi ac$. Then, a solution based on a scalar potential for the field \mathbf{H} rather than \mathbf{B} [3] quickly leads to the result that $\mathbf{B}(r < a) = 8\pi\mathbf{M}/3 = 2\mathbf{m}$, where the total dipole moment $\mathbf{m} = 4\pi a^3\mathbf{M}/3 = Qa^2\boldsymbol{\omega}/3c$ is the same as that calculated in (12). Again, the magnetic field outside the sphere is that due to a point dipole of strength \mathbf{m} .

³In many examples of electrodynamics of rotating systems we are less interested in the fields observed in the rotating frame than those in the lab frame. Yet, it may be that knowledge of some aspects of the fields in the moving frame is helpful in reaching an understanding of the fields in the lab frame. In this case it is often convenient to characterize the fields in the moving frame via the use of Lorentz transformations between the lab frame and an inertial (nonrotating) frame in which some point of the rotating system is at rest. For reviews of this approach, see [4, 5]. A pedagogic example is given in [6].

⁴This case is discussed most thoroughly by Ridgely [9, 10], but primarily for the interesting limit of steady charge and current distributions.

where quantities in observed in the rotating frame are labeled with a $'$. The transformations of charge and current density are

$$\rho' = \rho, \quad \mathbf{J}' = \mathbf{J} - \rho \mathbf{v}, \quad (16)$$

where \mathbf{v} ($v \ll c$) is the velocity with respect to the lab frame of the observer in the rotating frame. The transformations of the electromagnetic fields are

$$\mathbf{B}' = \mathbf{B}, \quad \mathbf{D}' = \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H}, \quad \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \quad \mathbf{H}' = \mathbf{H}. \quad (17)$$

The transformations of the electric and magnetic polarizations are

$$\mathbf{P}' = \mathbf{P} - \frac{\mathbf{v}}{c} \times \mathbf{M}, \quad \mathbf{M}' = \mathbf{M}, \quad (18)$$

if we regard these polarizations as defined by $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and $\mathbf{B}' = \mathbf{H}' + 4\pi\mathbf{M}'$.

The lab-frame bound charge and current densities $\rho_{\text{bound}} = -\nabla \cdot \mathbf{P}$ and $\mathbf{J}_{\text{bound}} = \partial\mathbf{P}/\partial t + c\nabla \times \mathbf{M}$ transform to

$$\rho'_{\text{bound}} = -\nabla' \cdot \mathbf{P}' - \frac{2\boldsymbol{\omega}' \cdot \mathbf{M}'}{c} + \frac{\mathbf{v}}{c} \cdot \nabla' \times \mathbf{M}', \quad (19)$$

$$\mathbf{J}'_{\text{bound}} = \frac{\partial\mathbf{P}'}{\partial t'} + c\nabla' \times \mathbf{M}' + \mathbf{v}(\nabla' \cdot \mathbf{P}') + \frac{\mathbf{v}}{c} \times \frac{\partial\mathbf{M}'}{\partial t'} + \boldsymbol{\omega}' \times \mathbf{P}' - \boldsymbol{\omega}' \frac{\partial\mathbf{P}'}{\partial\phi'}. \quad (20)$$

Force \mathbf{F} is invariant under the transformation (15). In particular, a charge q with velocity \mathbf{v}_q in the lab frame experiences a Lorentz force in the rotating frame given by

$$\mathbf{F}' = q \left(\mathbf{E}' + \frac{\mathbf{v}'_q}{c} \times \mathbf{B}' \right) = q \left(\mathbf{E} + \frac{\mathbf{v}_q}{c} \times \mathbf{B} \right) = \mathbf{F}, \quad (21)$$

where $\mathbf{v}'_q = \mathbf{v}_q - \mathbf{v}$. Similarly, the Lorentz force density \mathbf{f}' on charge and current densities in the rotating frame is

$$\mathbf{f}' = \rho' \mathbf{E}' + \frac{\mathbf{J}'}{c} \times \mathbf{B}' = (\rho'_{\text{free}} + \rho'_{\text{bound}}) \mathbf{E}' + \frac{\mathbf{J}'_{\text{free}} + \mathbf{J}'_{\text{bound}}}{c} \times \mathbf{B}'. \quad (22)$$

Maxwell's equations in the rotating frame can be written

$$\nabla' \cdot \mathbf{B}' = 0, \quad (23)$$

$$\nabla' \cdot \mathbf{D}' = 4\pi\rho'_{\text{free,total}} = 4\pi(\rho'_{\text{free}} + \rho'_{\text{other}}), \quad (24)$$

$$\nabla' \times \mathbf{E}' + \frac{\partial\mathbf{B}'}{\partial ct'} = 0, \quad (25)$$

$$\nabla' \times \mathbf{H}' - \frac{\partial\mathbf{D}'}{\partial ct'} = \frac{4\pi}{c} \mathbf{J}'_{\text{free,total}} = \frac{4\pi}{c} (\mathbf{J}'_{\text{free}} + \mathbf{J}'_{\text{other}}), \quad (26)$$

where $\rho'_{\text{free}} = \rho_{\text{free}}$ and $\mathbf{J}'_{\text{free}} = \mathbf{J}_{\text{free}} - \rho_{\text{free}}\mathbf{v}$ are the free charge and current densities, and the ‘‘other’’ charge and current densities that appear to an observer in the rotating frame are

$$\rho'_{\text{other}} = -\frac{\mathbf{v} \cdot \mathbf{J}'_{\text{free}}}{c^2} + \frac{\boldsymbol{\omega}' \cdot \mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c} \cdot \frac{\partial\mathbf{D}'}{\partial ct'}, \quad (27)$$

$$\mathbf{J}'_{\text{other}} = \rho'_{\text{free}}\mathbf{v} + \boldsymbol{\omega}' \times \frac{\mathbf{D}'}{4\pi} - \frac{\boldsymbol{\omega}' \partial\mathbf{D}'}{4\pi \partial\phi'} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial\mathbf{H}'}{\partial t'}. \quad (28)$$

The “other” charge and current distributions are sometimes called “fictitious” [1], but we find this term ambiguous. For an example with an “other” charge density $\boldsymbol{\omega}' \cdot \mathbf{H}'/2\pi c$ in the rotating frame, see [11].

Maxwell’s equations can also be expressed only in terms of the fields \mathbf{E}' and \mathbf{B}' and charge and current densities associated with free charges as well as with electric and magnetic polarization:

$$\nabla' \cdot \mathbf{E}' = 4\pi\rho'_{\text{total}}, \quad (29)$$

and

$$\nabla' \times \mathbf{B}' - \frac{\partial \mathbf{E}'}{\partial ct'} = \frac{4\pi}{c} \mathbf{J}'_{\text{total}}, \quad (30)$$

where

$$\begin{aligned} \rho'_{\text{total}} &= \rho'_{\text{free}} - \frac{\mathbf{v}}{c^2} \cdot \mathbf{J}'_{\text{free}} - \nabla' \cdot \mathbf{P}' + \frac{\boldsymbol{\omega}' \cdot \mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c} \cdot \frac{\partial \mathbf{D}'}{\partial ct'} \\ &= \rho'_{\text{free, total}} - \nabla' \cdot \mathbf{P}' \\ &= \rho'_{\text{free}} + \rho'_{\text{bound}} + \rho'_{\text{more}}, \end{aligned} \quad (31)$$

$$\rho'_{\text{more}} = -\frac{\mathbf{v}}{c^2} \cdot \left(\mathbf{J}'_{\text{free}} + \frac{\partial \mathbf{P}'}{\partial t'} + c\nabla' \times \mathbf{M}' \right) + \frac{\boldsymbol{\omega}' \cdot \mathbf{B}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c} \cdot \frac{\partial \mathbf{E}'}{\partial ct'}, \quad (32)$$

$$\begin{aligned} \mathbf{J}'_{\text{total}} &= \mathbf{J}'_{\text{free}} + \frac{\partial \mathbf{P}'}{\partial t'} + c\nabla' \times \mathbf{M}' + \rho'_{\text{free}} \mathbf{v} + \boldsymbol{\omega}' \times \frac{\mathbf{D}'}{4\pi} - \frac{\boldsymbol{\omega}'}{4\pi} \frac{\partial \mathbf{D}'}{\partial \phi'} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{H}'}{\partial t'} \\ &= \mathbf{J}'_{\text{free, total}} + \frac{\partial \mathbf{P}'}{\partial t'} + c\nabla' \times \mathbf{M}' \\ &= \mathbf{J}'_{\text{free}} + \mathbf{J}'_{\text{bound}} + \mathbf{J}'_{\text{more}}, \end{aligned} \quad (33)$$

$$\begin{aligned} \mathbf{J}'_{\text{more}} &= \mathbf{v} \left(\rho'_{\text{free}} - \nabla' \cdot \mathbf{P}' - \frac{2\boldsymbol{\omega}' \cdot \mathbf{M}'}{c} + \frac{\mathbf{v}}{c} \cdot \nabla' \times \mathbf{M}' \right) \\ &\quad + \boldsymbol{\omega}' \times \frac{\mathbf{E}'}{4\pi} - \frac{\boldsymbol{\omega}'}{4\pi} \frac{\partial \mathbf{E}'}{\partial \phi'} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{B}'}{\partial t'}. \end{aligned} \quad (34)$$

The contribution of the polarization densities to the source terms in Maxwell’s equations is much more complex in the rotating frame than in the lab frame. Because of the “other” source terms that depend on the fields in the rotating frame, Maxwell’s equations cannot be solved directly in this frame. Rather, an iterative approach is required in general.

The constitutive equations for linear isotropic media at rest in the rotating frame are

$$\mathbf{D}' = \epsilon \mathbf{E}', \quad \mathbf{B}' = \mu \mathbf{H}' - (\epsilon\mu - 1) \frac{\mathbf{v}}{c} \times \mathbf{E}', \quad (35)$$

in the rotating frame, and

$$\mathbf{D} = \epsilon \mathbf{E} + (\epsilon\mu - 1) \frac{\mathbf{v}}{c} \times \mathbf{H}, \quad \mathbf{B} = \mu \mathbf{H} - (\epsilon\mu - 1) \frac{\mathbf{v}}{c} \times \mathbf{E}, \quad (36)$$

in the lab frame. The lab-frame constitutive equations (36) are the same as for a nonrotating medium that moves with constant velocity \mathbf{v} with respect to the lab frame.

We can also write the constitutive equations (35) for a linear isotropic medium in terms of the fields \mathbf{B}' , \mathbf{E}' , \mathbf{P}' and \mathbf{M}' by noting that $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and $\mathbf{H}' = \mathbf{B}' - 4\pi\mathbf{M}'$, so that

$$\begin{aligned}\mathbf{P}' &= \frac{\epsilon - 1}{4\pi}\mathbf{E}', \\ \mathbf{M}' &= \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}'}{4\pi} - \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{E}'}{4\pi} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}'}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)}\frac{\mathbf{v}}{c} \times \mathbf{P}'.\end{aligned}\quad (37)$$

Similarly, the constitutive equations (36) in the lab frame can be written to order v/c as

$$\begin{aligned}\mathbf{P} &= \frac{\epsilon - 1}{4\pi}\mathbf{E} + \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{B}}{4\pi} = \frac{\epsilon - 1}{4\pi}\mathbf{E} + \frac{\epsilon\mu - 1}{\mu - 1}\frac{\mathbf{v}}{c} \times \mathbf{M}, \\ \mathbf{M} &= \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}}{4\pi} - \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{E}}{4\pi} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)}\frac{\mathbf{v}}{c} \times \mathbf{P}.\end{aligned}\quad (38)$$

Ohm's law for the conduction current \mathbf{J}_C has the same form for a medium with velocity \mathbf{u}' relative to the rotating frame as it does for a medium with velocity \mathbf{u} relative to the lab frame,

$$\mathbf{J}'_C = \sigma \left(\mathbf{E}' + \frac{\mathbf{u}'}{c} \times \mathbf{B}' \right) = \sigma \left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B} \right) = \mathbf{J}_C, \quad (39)$$

where σ is the electric conductivity of a medium at rest.

2.2.2 Fields of a Rotating Shell of Charge in a Rotating Frame

While the source charge and current densities ρ' and \mathbf{J}' are readily transformed from the lab frame to the rotating frame using eq. (16), it is not straightforward to solve Maxwell's equations in the rotating frame because the "other" charge and current densities in eqs. (31)-(34) depend on the as-yet-unknown fields in that frame.

However, in the present example, we can simply transform the fields (1) and (14) to the rotating frame using eq. (17), with the results

$$\begin{aligned}\mathbf{E}'(r < a) &= \mathbf{E}(r < a) + \frac{\mathbf{v}}{c} \times \mathbf{B}(r < a) = \frac{\mathbf{v}}{c} \times \frac{2\mathbf{m}}{a^3}, \\ \mathbf{E}'(r > a) &= \mathbf{E}(r > a) + \frac{\mathbf{v}}{c} \times \mathbf{B}(r > a) = \frac{Q}{r^2}\hat{\mathbf{r}} + \frac{\mathbf{v}}{c} \times \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3},\end{aligned}\quad (40)$$

$$\begin{aligned}\mathbf{B}'(r < a) &= \mathbf{B}(r < a) = \frac{2\mathbf{m}}{a^3}, \\ \mathbf{B}'(r > a) &= \mathbf{B}(r > a) = \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}.\end{aligned}\quad (41)$$

where $\mathbf{v} = \boldsymbol{\omega}' \times \mathbf{r}$. Even when $\boldsymbol{\omega}' = \boldsymbol{\omega}$ so that the electric charge appears at rest in the rotating frame, a nonzero magnetic field (41) is observed in that frame, which can be attributed to the "fictitious" currents.

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