The Fields of a Pulsed, Small Dipole Antenna

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1 Problem

Deduce the electromagnetic fields of a small dipole antenna whose time-dependent electric dipole moment $\mathbf{p}(t)$ is given.

2 Solution

Analytic descriptions of the fields of pulsed, finite-sized antennas are only approximate. For a review, see [1]. In general, numerical simulations are required for detailed characterization of the fields of pulsed antennas. See, for example, [2]-[6]. Here, we note that an "exact" form can be given for the fields in both the near and far zones of a pulsed dipole antenna in the limit that the size of the antenna is vanishingly small.

One approach to a solution is based on the well-known electromagnetic fields of an oscillating "point" electric dipole located at the origin, whose dipole moment is,

$$\mathbf{p}(t) = \mathbf{p}_0 \, e^{-i\omega t}. \tag{1}$$

In Gaussian units (and in vacuum), these fields can be written as, [7]

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \left(-\frac{\omega^2}{c^2 r} \left((\mathbf{p}_0 \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}} \right) + \left(-\frac{i\omega}{c\, r^2} + \frac{1}{r^3} \right) (3(\mathbf{p}_0 \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}} - \mathbf{p}_0) \right) e^{i(kr-\omega t)}, && (2) \\ \mathbf{B}(\mathbf{r}, t) &= -\left(\frac{\omega^2}{c^2 r} + \frac{i\omega}{c\, r^2} \right) \mathbf{p}_0 \times \hat{\mathbf{r}}\, e^{i(kr-\omega t)}. && (3) \end{aligned}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector from the center of the dipole to the observer, and c is the speed of light. We recall that these fields are functions of the dipole moment evaluated at the retarded time,

$$t' = t - r/c, \tag{4}$$

which contains the insight that changes in the state of the dipole cause changes in the fields which propagate with the speed of light. We write the retarded dipole moment as,

$$[\mathbf{p}] = \mathbf{p}(t') = \mathbf{p}(t - r/c) = \mathbf{p}_0 \, e^{-i\omega(t-r/c)} = \mathbf{p}_0 \, e^{-i(kr-\omega t)}, \tag{5}$$

so that its retarded time derivatives are,

$$[\dot{\mathbf{p}}] = \frac{d\mathbf{p}(t')}{dt} = -i\omega \mathbf{p}_0 \, e^{-i(kr-\omega t)}, \qquad [\ddot{\mathbf{p}}] = \frac{d^2\mathbf{p}(t')}{dt^2} = -\omega^2 \mathbf{p}_0 \, e^{-i(kr-\omega t)}. \tag{6}$$

Writing eqs. (2)-(3) in terms of the retarded quantities (5)-(6) we obtain the desired forms for the fields of a "point" electric dipole with arbitrary time dependence,

$$
\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \frac{([\ddot{\mathbf{p}}]\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{c^2 r} + \frac{3([\dot{\mathbf{p}}]\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - [\dot{\mathbf{p}}]}{c\,r^2} + \frac{3([\mathbf{p}]\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - [\mathbf{p}]}{r^3}\,, && (7)\\
\mathbf{B}(\mathbf{r},t) &= \frac{[\ddot{\mathbf{p}}]\times\hat{\mathbf{r}}}{c^2 r} + \frac{[\dot{\mathbf{p}}]\times\hat{\mathbf{r}}}{c\,r^2}\,. && (8)
\end{aligned}
$$

The fields (7)-(8) can also be found by a straightforward but lengthy calculation of the retarded potentials of a "point" dipole with arbitrary time dependence, followed by calculation of the fields from the potentials. See, for example, sec. 7.1 of [5], sec. 2.2.3 of [8] (which uses the Hertz potential), and [9]. A version of this argument is given in the Appendix below.

Another approach is to use the general expressions for the electromagnetic fields in terms of retarded charge- and current-density distributions, $[\varrho]$ *and* $[\mathbf{J}]$*, respectively [10, 11],*

$$
\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \int \frac{[\varrho]\,\hat{\mathbf{r}}}{r^2}\,d\mathrm{Vol}' + \int \frac{([\mathbf{J}]\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} + ([\mathbf{J}]\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{c\,r^2}\,d\mathrm{Vol}' + \int \frac{([\dot{\mathbf{J}}]\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{c^2 r}\,d\mathrm{Vol}'\\
&= \int \frac{[\varrho]\,\hat{\mathbf{r}}}{r^2}\,d\mathrm{Vol}' + \int \frac{2([\mathbf{J}]\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - [\mathbf{J}]}{c\,r^2}\,d\mathrm{Vol}' + \int \frac{([\dot{\mathbf{J}}]\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{c^2 r}\,d\mathrm{Vol}', && (9)
\end{aligned}
$$

and,

$$
\mathbf{B}(\mathbf{r},t) = \int \frac{[\mathbf{J}]\times\hat{\mathbf{r}}}{c\,r^2}\,d\mathrm{Vol}' + \int \frac{[\dot{\mathbf{J}}]\times\hat{\mathbf{r}}}{c^2 r}\,d\mathrm{Vol}', \qquad (10)
$$

where $\hat{\mathbf{r}} = \mathbf{r}/r = (\mathbf{x}-\mathbf{x}')/\left|\mathbf{x}-\mathbf{x}'\right|$*, and quantities inside brackets, [...], are evaluated at the retarded time* $t' = t - r/c$*. In the case of a time-dependent "point" dipole* $\mathbf{p}(t)$*, it seems natural to identify the time derivative* $\dot{\mathbf{p}}$ *with the current element* $\mathbf{J}\,d\mathrm{Vol}$*, and* $\ddot{\mathbf{p}}$ *with* $\dot{\mathbf{J}}\,d\mathrm{Vol}$*. Then, we expect that the first term of eq. (9) gives the retarded electric dipole fields, and expressions (9)-(10) become,*

$$
\mathbf{E}(\mathbf{r},t) = \frac{3([\mathbf{p}]\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - [\mathbf{p}]}{r^3} + \frac{2([\dot{\mathbf{p}}]\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - [\dot{\mathbf{p}}]}{c\,r^2} + \frac{([\ddot{\mathbf{p}}]\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{c^2 r}\,, \qquad (11)
$$

and,

$$
\mathbf{B}(\mathbf{r},t) = \frac{[\dot{\mathbf{p}}]\times\hat{\mathbf{r}}}{c\,r^2} + \frac{[\ddot{\mathbf{p}}]\times\hat{\mathbf{r}}}{c^2 r}\,. \qquad (12)
$$

These fields are the same as eqs. (7)-(8) except for the factor of 2 rather than 3 in the second term of eq. (11). Apparently it is rather subtle to explain how the proper use of eq. (9) for a "point" dipole corrects this discrepancy [12, 13]. So, we content ourselves that the first solution gave the correct result fairly quickly.

2.1 Oscillating Dipole

2.1.1 The Dipole Oscillates Only at $t > 0$

A simple example of the use of eqs. (7)-(8) is for the case that a small dipole antenna oscillates according to eq. (1) only for $t > 0$, with no oscillation for $t < 0$. Then, at an observation point $\mathbf{r}$, the fields vanish for $t < r/c$, and have the form (2)-(3) for $t > r/c$.

The onset of oscillation of the dipole at time $t = 0$ can be considered as a signal to the observer, which signal can first be detected via nonzero fields at the position of the observer at time $T = r/c$. Hence, the signal velocity is $T/r = c$ as expected. This conclusion holds for any distance r, whether in the near or far zone of the dipole antenna.[1]

2.1.2 Steady Oscillations

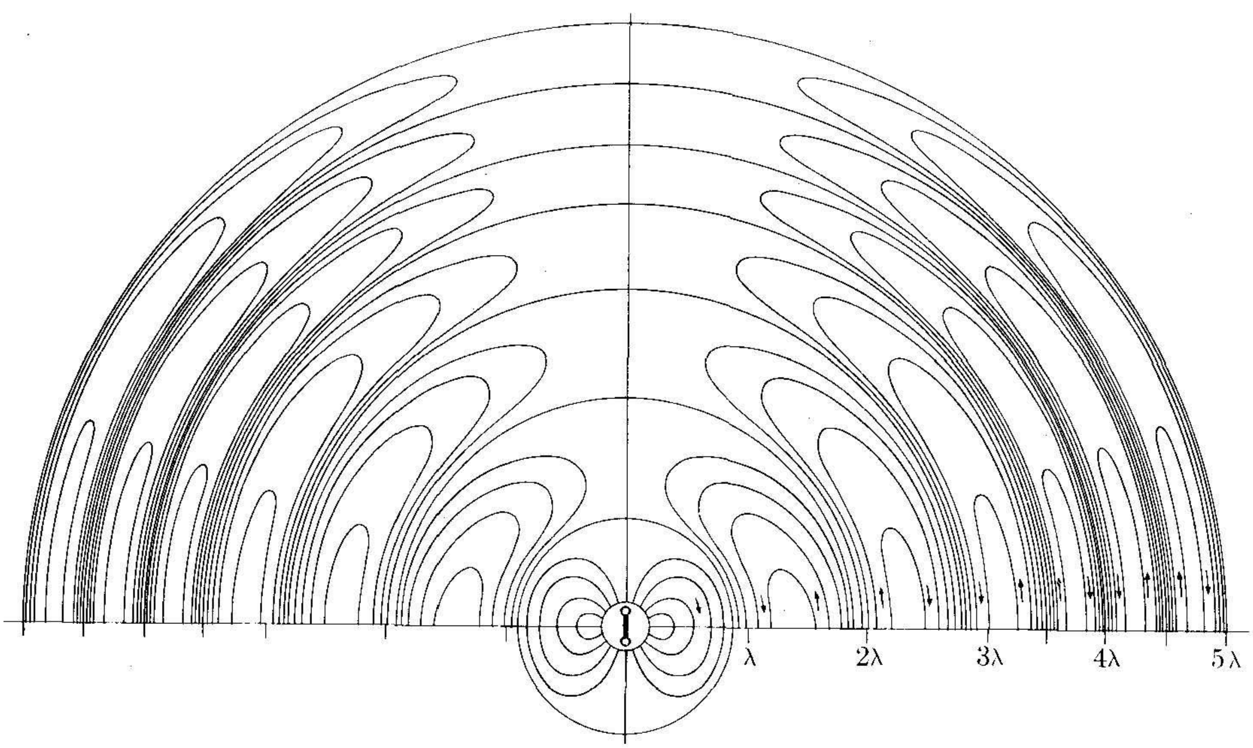
An instructive plot of the electric field of a "point" oscillating dipole has been given in sec. 14-7 of [10], as shown on the next page.

In the far zone ($r \gg \lambda$) spherical surfaces of zero electric field are spaced every half wavelength, and propagate radially at velocity c. However, in the near zone ($r \lesssim \lambda$), these spherical surfaces propagate radially at a velocity in excess of c.

This behavior is not evidence of superluminal signal propagation, because these spherical surfaces, which exist in the steady field pattern at a single frequency of oscillation, are not associated with any modulation of the source of the fields. When, the source is modulated so as to generate a signal, as in examples 1 and 2, the signal propagates at velocity c at any distance from the source.

See sec. 4 of [15] for further discussion of phase and group velocity of the fields of an oscillating dipole.

Remark: An extension of the present problem to the case that the antenna is immersed in a dissipative medium (such as water) has been given in [16].



[1]That is, there is no superluminal signal propagation in the near field of a dipole antenna, contrary to a recent claim [14].

2.1.3 The Dipole Oscillates for Only 1/2 Cycle

Suppose the electric dipole moment has the time dependence,

$$\mathbf{p}(t) = p_0\,\hat{\mathbf{z}} \begin{cases} 0 & (t < 0), \\ \sin\omega t & (0 \le t \le \pi/\omega), \\ 0 & (t > \pi/\omega). \end{cases} \tag{13}$$

Then for an observer at distance r from the source the electromagnetic fields are nonzero only during the interval $0 < t' = t - r/c < \pi/\omega$, when the retarded moment and its derivatives are,

$$[\mathbf{p}] = p_0\,\hat{\mathbf{z}}\sin\omega t', \qquad [\dot{\mathbf{p}}] = \omega p_0\,\hat{\mathbf{z}}\cos\omega t', \qquad [\ddot{\mathbf{p}}] = -\omega^2 p_0\,\hat{\mathbf{z}}\sin\omega t', \tag{14}$$

where $k = \omega/c$ is the wave number, such that the fields follow from eqs. (7)-(8) as,

$$\mathbf{E}(\mathbf{r},t) = -p_0\frac{k^2\sin\omega t'}{r}\sin\theta\,\hat{\boldsymbol{\theta}} + p_0\left(\frac{k\cos\omega t'}{r^2} + \frac{\sin\omega t'}{r^3}\right)(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}), \tag{15}$$

$$\mathbf{B}(\mathbf{r},t) = p_0\left(-\frac{k^2\sin\omega t'}{r} + \frac{k\cos\omega t'}{r^2}\right)\sin\theta\,\hat{\boldsymbol{\phi}}. \tag{16}$$

The radial flow of energy is described by the radial component of the Poynting vector $\mathbf{S}$,

$$\begin{aligned} S_r &= \frac{c}{4\pi}\hat{\mathbf{r}}\cdot\mathbf{E}\times\mathbf{B} \\ &= \frac{ck^6p_0^2\sin^2\theta}{4\pi}\left[\frac{\sin^2\omega t'}{(kr)^2} + \left(\frac{1}{2(kr)^5} - \frac{1}{(kr)^3}\right)\sin 2\omega t' + \frac{\cos 2\omega t'}{(kr)^4}\right]. \end{aligned} \tag{17}$$

There is also a meridional flow of energy,

$$\begin{aligned} S_\theta &= \frac{c}{4\pi}\hat{\boldsymbol{\theta}}\cdot\mathbf{E}\times\mathbf{B} \\ &= \frac{ck^6p_0^2\sin 2\theta}{4\pi}\left[\left(\frac{1}{(kr)^5} - \frac{1}{(kr)^3}\right)\frac{\sin 2\omega t'}{2} + \frac{\cos 2\omega t'}{(kr)^4}\right]. \end{aligned} \tag{18}$$

Associated with this flow of energy is the density u of electromagnetic field energy,

$$\begin{aligned} u &= \frac{E^2 + B^2}{8\pi} \\ &= \frac{k^6p_0^2\sin^2\theta}{4\pi}\left[\frac{\sin^2\omega t'}{(kr)^2} - \left(\frac{3}{(kr)^5} - \frac{1}{(kr)^3}\right)\frac{\sin 2\omega t'}{2} - \frac{\cos^2\omega t'}{(kr)^4} - \frac{3\sin^2\omega t'}{2(kr)^6}\right] \\ &\quad + \frac{k^6p_0^2}{2\pi}\left[\frac{\cos^2\omega t'}{(kr)^4} + \frac{\sin 2\omega t'}{(kr)^5} + \frac{\sin^2\omega t'}{(kr)^6}\right], \end{aligned} \tag{19}$$

which obeys $\boldsymbol{\nabla}\cdot\mathbf{S} = \partial u/\partial t$.

The analysis that has led to eq. (17) supposes that electromagnetic effects propagate at the speed of light, so the flow of energy can be regarded as the product of an energy density and a velocity vector whose magnitude is c. However, only the first term in the radial Poynting vector is the product of c and a term (the first) in the energy density u. In general, only part of the energy density u at a point in space is flowing, and the rest is (temporarily) stored at that point.

The classical model of this flow and temporary storage of energy is rather complicated, with behavior anticipated by Huygens. In the vicinity of any point in space there can be stored electromagnetic energy that changes with time due to both the absorption of energy in transit from elsewhere, and the emission of some of the stored energy. The emission of energy is in all directions, although inward emission is weaker at larger radius, and vanishes at very large distances according to eq. (17).

In the first part of a half cycle of retarded time t' the flow of energy is largely outwards, and exceeds the first term of eq. (17). During the second quarter cycle the outward flow of energy at finite r is less than the first term of eq. (17), and for small r the flow of energy is inward. One (common) interpretation is that the flow described by the first term of eq. (17) if physically distinct from that of the other terms, and only this term should be called "radiation". I have come to prefer the view that the Poynting vector should not be partitioned (which is always awkward for a quadratic function), and that it should be identified with the concept of "radiation" [17]. In this view some of the energy that flows from the source during the first quarter cycle becomes stored in the fields, and later is emitted to join the outward or inward flow of energy as then obtains in the local vicinity. The "radiation" per solid angle grows with radius in the second quarter cycle, and approaches the value of the "radiation to infinity" (given by the first term of eq. (17)) only at large r. This behavior is more dramatically illustrated in sec. 2.4.2 (and also in sec. 2.5).

Another variant is for the dipole moment to rise sinusoidally to p_0 over a quarter cycle, after which it remains constant for a finite time interval before returning to zero over a second quarter cycle. Energy flows only during retarded times corresponding to the two quarter cycles, while between these intervals the electrostatic energy of dipole $\mathbf{p}_0$ is stored in space. The source of the "radiation" emitted during the second quarter cycle is the stored field energy, noting that energy flows back onto the dipole during the entire second quarter cycle.

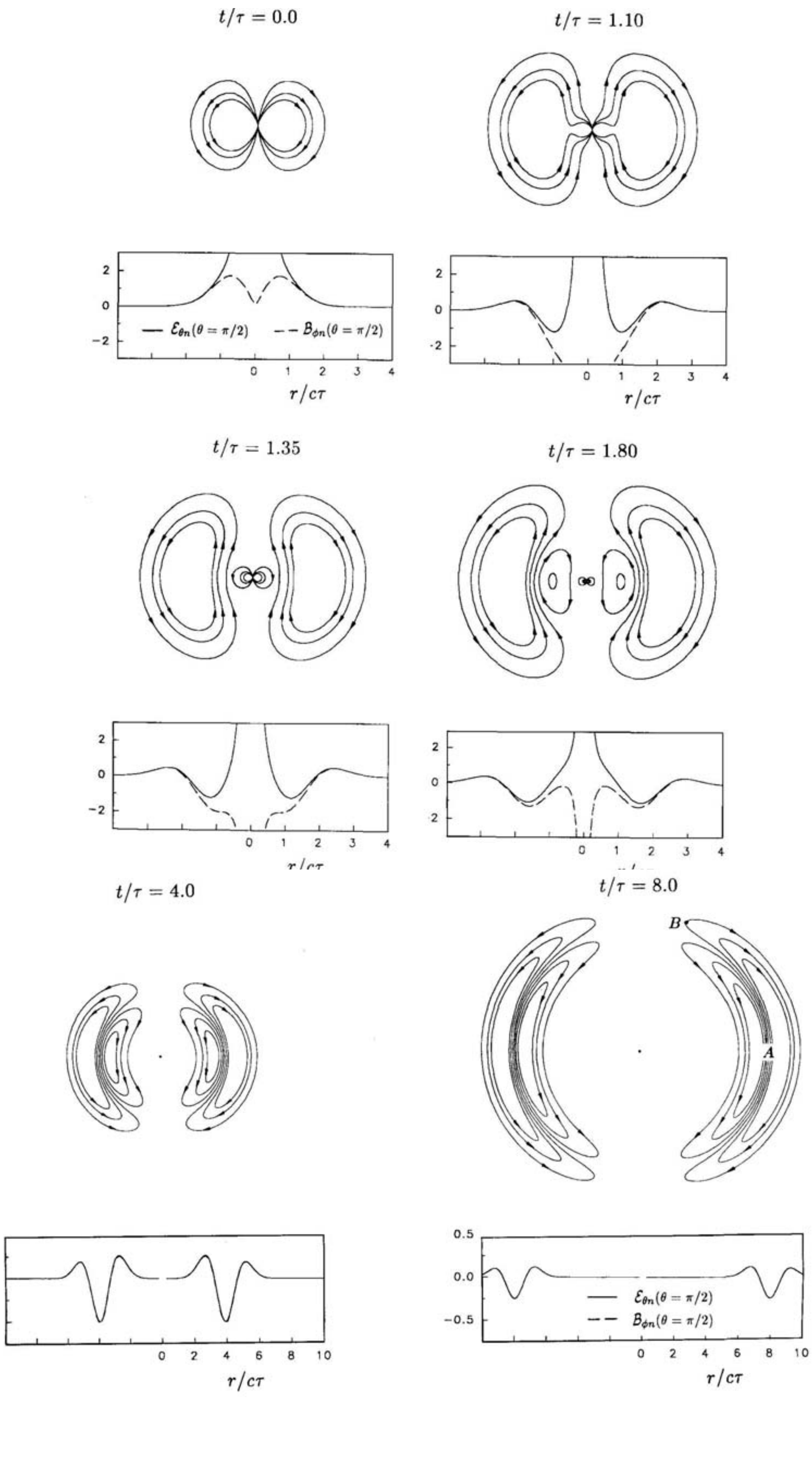
2.2 The Dipole Moment Has Gaussian Time Dependence

Plots of the electric field for a dipole moment with Gaussian time dependence,

$$\mathbf{p} = p_0 e^{-t^2/\tau^2}\,\hat{\mathbf{z}}, \tag{20}$$

have been given in sec. 7.1 of [5], as shown on the following page.

To identify a signal velocity associated with this pulse, note that the long tail of positive electric field at early times cannot be called a signal. Rather, the narrow pulse of negative electric field centered at radial position $r = c\,t$ is more properly called the signal. The signal velocity is then c.


$t/\tau = 0.0$
$t/\tau = 1.10$
$\mathcal{E}_{\theta n}(\theta = \pi/2)$
$B_{\phi n}(\theta = \pi/2)$
$r/c\tau$
$r/c\tau$
$t/\tau = 1.35$
$t/\tau = 1.80$
$t/\tau = 4.0$
$t/\tau = 8.0$
B
A
$r/c\tau$
$\mathcal{E}_{\theta n}(\theta = \pi/2)$
$B_{\phi n}(\theta = \pi/2)$
$r/c\tau$

2.3 Dipole Whose Moment Varies Quadratically with Time

J. Heras draws our attention to the special case of a "point" dipole at the origin whose moment $\mathbf{p}$ varies quadratically with time [18],

$$\mathbf{p}(t) = \left(p_0 + \dot{p}_0 t + \frac{\ddot{p}_0 t^2}{2} \right) \hat{\mathbf{z}} \equiv p(t)\, \hat{\mathbf{z}}. \tag{21}$$

The retarded moments and their derivatives associated with an observer at (r, θ, ϕ) in spherical coordinates are,

$$\begin{aligned}
[\mathbf{p}] &= \left(p_0 + \dot{p}_0 (t - r/c) + \frac{\ddot{p}_0 (t^2 - 2rt/c + r^2/c^2)}{2} \right) \hat{\mathbf{z}} \\
&= \left(p(t) - \frac{r}{c}\dot{p}(t) + \frac{r^2}{2c^2}\ddot{p}_0 \right) \hat{\mathbf{z}} = \mathbf{p}(t) - \frac{r}{c}\dot{\mathbf{p}}(t) + \frac{r^2}{2c^2}\ddot{\mathbf{p}}, \qquad (22) \\
[\dot{\mathbf{p}}] &= (\dot{p}_0 + \ddot{p}_0 (t - r/c))\, \hat{\mathbf{z}} = \left(\dot{p}(t) - \frac{r}{c}\ddot{p}_0 \right) \hat{\mathbf{z}} = \dot{\mathbf{p}}(t) - \frac{r}{c}\ddot{\mathbf{p}}, \qquad (23) \\
[\ddot{\mathbf{p}}] &= \ddot{p}_0\, \hat{\mathbf{z}} = \ddot{\mathbf{p}}. \qquad (24)
\end{aligned}$$

Then the fields at the observer follow from eqs. (7)-(8) as,

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= \frac{(\ddot{\mathbf{p}} \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}} - \ddot{\mathbf{p}}}{c^2 r} - \frac{3(\ddot{\mathbf{p}} \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}} - \ddot{\mathbf{p}}}{2c^2 r} + \frac{3(\mathbf{p}(t) \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}} - \mathbf{p}(t)}{r^3} \\
&= -\frac{\ddot{\mathbf{p}} + (\ddot{\mathbf{p}} \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}}}{2c^2 r} + \frac{3(\mathbf{p}(t) \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}(t)}{r^3} \\
&= \frac{\ddot{p}_0}{c^2 r} \left(-\cos\theta\, \hat{\mathbf{r}} + \frac{\sin\theta}{2}\, \hat{\boldsymbol{\theta}} \right) + \frac{p(t)}{r^3}(2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}), \qquad (25) \\
\mathbf{B}(\mathbf{r}, t) &= \frac{\ddot{\mathbf{p}}(t) \times \hat{\mathbf{r}}}{c^2 r} + \frac{(\dot{\mathbf{p}}(t) - r\, \ddot{\mathbf{p}}(t)/c) \times \hat{\mathbf{r}}}{c\, r^2} = \frac{\dot{\mathbf{p}}(t) \times \hat{\mathbf{r}}}{c\, r^2} = \frac{\dot{p}(t) \sin\theta}{c\, r^2}\, \hat{\boldsymbol{\phi}}. \qquad (26)
\end{aligned}$$

Because of the special character of the time dependence of the source, the fields at a distant observer at time t can be written in terms of source quantities at that time. However, this does not imply instantaneous propagation of the fields.

The flow of energy at the observer at time t is described by the Poynting vector,

$$\mathbf{S}(\mathbf{r}, t) = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} = \left(\frac{\ddot{p}_0}{2c^2} + \frac{p(t)}{r^2} \right) \frac{\dot{p}(t) \sin^2\theta}{4\pi r^3}\, \hat{\mathbf{r}} + \left(\frac{\ddot{p}_0}{c^2} - \frac{2p(t)}{r^2} \right) \frac{\dot{p}(t) \cos\theta \sin\theta}{4\pi r^3}\, \hat{\boldsymbol{\theta}}. \tag{27}$$

The power that crosses a sphere of radius r at time t is,

$$P(r, t) = \int r^2 \mathbf{S} \cdot \hat{\mathbf{r}}\, d\Omega = \frac{2\dot{p}(t)}{3r} \left(\frac{\ddot{p}_0}{2c^2} + \frac{p(t)}{r^2} \right). \tag{28}$$

While some of the energy flow across radius r becomes stored in the fields at larger radii, some energy flows to "infinity" at the speed of light. Energy emitted by the dipole at source time t_s arrives at radius r at time $t = t_s + r/c$. Evaluating eq. (28) at this time, and keeping only the leading terms, we find that,

$$\begin{aligned}
P(r, t = t_s + r/c) &= \frac{2[\dot{p}_0 + \ddot{p}_0 (t_s + r/c)]}{3r} \left(\frac{\ddot{p}_0}{2c^2} + \frac{p_0 + \dot{p}_0 (t_s + r/c) + \ddot{p}_0 (t_s + r/c)^2/2}{r^2} \right) \\
&\approx \frac{2\ddot{p}_0^2}{3c^2}, \qquad (29)
\end{aligned}$$

in agreement with the Larmor formula,

$$P = \frac{2\left|[\ddot{\mathbf{p}}]\right|^2}{3c^3}, \tag{30}$$

for the power "radiated to infinity" by a time-dependent electric dipole.[2] See, for example. eq. (67.8) of [19].

2.4 Dipole Whose Moment Varies Quadratically with Time for $0 < t < t_0$

It is instructive to consider the case where the dipole moment is zero for time $t < 0$, then rises quadratically with time according to,

$$\mathbf{p}(t) = \frac{\ddot{p}_0 t^2}{2}\,\hat{\mathbf{z}} \equiv p(t)\,\hat{\mathbf{z}}. \tag{31}$$

eq. (21) until time t_0, after which it remains constant at,

$$\mathbf{p}_f = \frac{\ddot{p}_0 t_0^2}{2}\,\hat{\mathbf{z}} \equiv p_f\,\hat{\mathbf{z}}. \tag{32}$$

The second time derivative $\ddot{p}$ is discontinuous at times $t = 0$ and t_0, and the derivative $\dot{p}$ is discontinuous at time $t = t_0$. We think of these effects as limiting cases of short pulses during which the dipole moment changes very rapidly.[3]

2.4.1 Effect of the Pulse at $t_{\rm source} = 0$

During the pulse at $t_{\rm source} = 0$ the dipole moment p and its time derivative $\dot{p}$ are zero, while its second derivative $\ddot{p}$ rises from zero to $\ddot{p}_0$ (and so has an average value of $\ddot{p}_0/2$). Then, according to eqs. (7)-(8) the electromagnetic fields associated with this pulse have only a $1/r$ dependence. However, these fields have a finite value during the infinitesimal duration of the pulse, such that no energy is emitted during this pulse in the limit that it has zero width.

2.4.2 Effect of the Pulse at $t_{\rm source} = t_0$

During the pulse at $t_{\rm source} = t_0$ the dipole moment remains at $p_f = \ddot{p}_0 t_0^2/2$, its time derivative drops from $\ddot{p}_0 t_0$ to zero and so has an average value of $\dot{p}(t_0) = \ddot{p}_0 t_0/2$. Further, we can say that the drop in $\dot{p}$ during the pulse is due to a delta function in $\ddot{p}$ at time t_0,

$$\dot{p}(t_0^+) = 0 = \dot{p}(t_0^-) + \int_{\text{pulse at } t_0} \ddot{p}(t)\,dt = \ddot{p}_0 t_0 \left(1 - \int_{\text{pulse at } t_0} \delta(t - t_0)\,dt\right), \tag{33}$$

and hence during the pulse at time t_0,

$$\ddot{p}(t) = -\ddot{p}_0 t_0 \delta(t - t_0). \tag{34}$$

[2]Thanks to David J. Griffiths for pointing this out, which is contrary to a claim in [18].
[3]Closely related problems have been considered in [20].

The fields from the source pulse at time t_0 arrive at the distant observer at time $t = t_0 + r_c$. According to that observer, the retarded source properties associated with these fields are just the source properties at time t_0 identified above. Thus, the fields associated by the observer with the pulse follow from eqs. (7)-(8) as,

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}, t = t_0 + r/c) &= -\frac{\ddot{p}_0 t_0 \delta(t - t_0 - r/c)}{c^2 r} \sin\theta\,\hat{\boldsymbol{\theta}} + \frac{\ddot{p}_0 t_0}{2r^3}\left(t_0 + \frac{r}{c}\right)(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \\
&\equiv \mathbf{E}_{1/r} + \mathbf{E}_{\text{other}}\,, \qquad (35) \\
\mathbf{B}(\mathbf{r}, t = t_0 + r/c) &= \left(-\frac{\ddot{p}_0 t_0\,\delta(t - t_0 - r/c)}{c^2 r} + \frac{\ddot{p}_0 t_0}{2c\,r^2}\right)\sin\theta\,\hat{\boldsymbol{\phi}}\,, \\
&\equiv \mathbf{B}_{1/r} + \mathbf{B}_{\text{other}}\,. \qquad (36)
\end{aligned}
$$

The first terms on the righthand sides of eqs. (35)-(36) fall off as $1/r$ and are associated with an intense pulse of "radiation to infinity". Note that the Poynting vector of this pulse of fields contains more than the term,

$$
\mathbf{S}_{1/r^2} = \frac{c}{4\pi}\mathbf{E}_{1/r} \times \mathbf{B}_{1/r}\,. \qquad (37)
$$

Indeed, we have that the radial component of the "other" Poynting vector,

$$
\mathbf{S}_{\text{other}} = \mathbf{S} - \mathbf{S}_{1/r^2} \qquad (38)
$$

is,

$$
S_{\text{other},r} = -\frac{\ddot{p}_0^2 t_0^2 \sin^2\theta\,\delta(t - t_0 - r/c)}{4\pi c\,r^4}\left(t_0 + \frac{r}{2c}\right) + \text{terms with no } \delta \text{ function.} \qquad (39)
$$

Thus, there is a correction to the pulse of "radiation to infinity" that is negative and weaker with distance. This means that the product $r^2 S_{1/r^2}$ grows with distance. In other words, more energy is "radiated to infinity" in the pulse than is emitted in the pulse at the source.

Similarly, the θ component of the "other" Poynting vector is,

$$
S_{\text{other},\theta} = \frac{\ddot{p}_0^2 t_0^2 \cos\theta\sin\theta\,\delta(t - t_0 - r/c)}{4\pi c^2 r^4}\left(t_0 + \frac{r}{c}\right) + \text{term with no } \delta \text{ function.} \qquad (40)
$$

This "sideways" flow of energy cannot be associated with energy emitted by the pulse in the source and subsequently transported radially to the observer.

To clarify these remarkable results, we consider the energy stored in the fields before and after the pulse at source time t_0.

2.4.3 Energy Stored in the Fields

At large times, after the pulse emitted at $t = t_0$ has passed, the magnetic field is zero and the electric field is that of a static dipole of moment $\mathbf{p}_f$,

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}, t > t_0 + r/c) &= \frac{3(\mathbf{p}_f \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}_f}{r^3} = \frac{\ddot{p}_0 t_0^2}{2r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}), \qquad (41) \\
\mathbf{B}(\mathbf{r}, t > t_0 + r/c) &= 0. \qquad (42)
\end{aligned}
$$

The density of energy stored in the electric field is,

$$u = \frac{E^2}{8\pi} = \frac{\dddot{p}_0^2 t_0^4}{32\pi r^6}(3\cos^2\theta + 1). \tag{43}$$

The total energy stored in the static field at $r > r_0$ is,

$$U_0(r_0) = 2\pi \int_{r_0}^{\infty} r^2\, dr \int_{-1}^{1} d\cos\theta\, u = \frac{\dddot{p}_0^2 t_0^4}{12 r_0^3}\,. \tag{44}$$

We might expect that the stored energy U_0 is due the energy emitted at the source between times $t = 0$ and t_0, which corresponds to energy crossing the surface r_0 at times $r_0/c < t < t_0 + r_0/c$. Recalling eq. (27), this energy is,

$$\begin{aligned} U_1(r_0) &= 2\pi r_0^2 \int_{r_0/c}^{t_0+r_0/c)} dt \int_{-1}^{1} d\cos\theta\, S_r(r_0,t) = \frac{2}{3r_0}\int_{r_0/c}^{t_0+r_0/c)} dt \left(\frac{p(t)\dot{p}(t)}{r_0^2} - \frac{\dot{p}(t)\ddot{p}_0}{2c^2}\right) \\ &= \frac{\dddot{p}_0^2}{3r_0}\int_{r_0/c}^{t_0+r_0/c)} dt \left(\frac{t^3}{r_0^2} - \frac{t}{c}\right) = \frac{\dddot{p}_0^2 t_0^4}{12 r_0^3} + \frac{\dddot{p}_0^2 t_0^3}{3c\, r_0^2} + \frac{\dddot{p}_0^2 t_0^2}{6 r_0}\,. \end{aligned} \tag{45}$$

The energy emitted between time $t = 0$ and t_0 and later stored in the fields at $r > r_0$ is greater than the final energy stored there. It must be that as the pulse emitted at time t_0 passes through the fields it "sweeps up" some of the energy stored there and carries it off to "infinity."

Indeed, from eq. (39) we learn that the when the pulse it at radius r_0 it transports less energy across that surface than it carries to "infinity" in the amount,

$$\begin{aligned} U_2 &= 2\pi r_0^2 \int_{-1}^{1} d\cos\theta \int_{\text{pulse}} dt\, S_{\text{other},r}(r_0,t) = -\frac{\dddot{p}_0^2 t_0^2}{4c\, r_0^2}\left(t_0 + \frac{r}{2c}\right)\int_{-1}^{1} d\cos\theta\, \sin^2\theta \\ &= -\frac{\dddot{p}_0^2 t_0^3}{3c\, r_0^2} - \frac{\dddot{p}_0^2 t_0^2}{6 r_0}\,. \end{aligned} \tag{46}$$

The energy that remains stored in the fields at $r > r_0$ after the passage of the pulse is,

$$U_1 + U_2 = \frac{\dddot{p}_0^2 t_0^4}{12 r_0^3} = U_0\,. \tag{47}$$

In this example "radiation to infinity" is emitted steadily during the interval $0 < t < t_0$ as described in sec. 2.3, followed by a pulse emitted at time t_0. But less energy was emitted by the source at that time than eventually flowed to "infinity". As the pulse passed through the fields at nonzero r it "swept up" some of the energy previously stored there. This is a kind of laser effect in the vacuum.

This classical effect anticipates the quantum effect that a state populated by n bosons has a higher amplitude to stimulate emission of another boson into that state than if the state were populated by $m < n$ bosons. See, for example, chap. 4, vol. III of [21].

For examples of this effect in the case of magnetic dipole radiation, see [22].

2.5 Dipole Whose Moment Decays Exponentially

We now consider the case of exponential decay of an electric dipole moment from its constant value at $t < 0$,

$$\mathbf{p}(t) = p_0\, \hat{\mathbf{z}} \begin{cases} 1 & (t < 0), \\ e^{-t/t_0} & (t > 0). \end{cases} \tag{48}$$

This behavior is a classical model of the decay of an excited atom, and also is of relevance to spark discharges.

Similar to the discussion at the beginning of sec. 2.5.3, at distance r from the dipole and for times $t < r/c$ the magnetic field is zero, the electric field is,

$$\mathbf{E}_0(\mathbf{r}) = \frac{3(\mathbf{p}_0 \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}_0}{r^3} = \frac{p_0(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})}{r^3} \qquad (t < r/c), \tag{49}$$

and the electric field energy stored outside a sphere of radius $r_0 \gg a$ at times $t < r_0/c$ is,

$$U_0 = \int_{r>r_0} \frac{E_0^2}{8\pi}\, d^3\mathbf{x} = 2\pi \int_{r_0}^{\infty} r^2\, dr \int_{-1}^{1} d\cos\theta\, \frac{p_0^2(3\cos^2\theta + 1)}{8\pi r^6} = \frac{p_0^2}{3r_0^3}\,. \tag{50}$$

After time $t = r/c$ the field and the stored energy at radius r change, and at large times they are both zero. Some of the stored energy returns to the source, and some is "radiated to infinity."

To clarify this, we note that according to an observer at radius r at time $t > r/c$ the retarded electric dipole moment are its derivatives are,

$$[\mathbf{p}] = p_0\, e^{-(t-r/c)/t_0}\,\hat{\mathbf{z}}, \qquad [\dot{\mathbf{p}}] = -\frac{p_0}{t_0}\, e^{-(t-r/c)/t_0}\,\hat{\mathbf{z}}, \qquad [\ddot{\mathbf{p}}] = \frac{p_0}{t_0^2}\, e^{-(t-r/c)/t_0}\,\hat{\mathbf{z}}, \tag{51}$$

so the fields for $t > r/c$ follow from eqs. (7)-(8) as,

$$\mathbf{E}(\mathbf{r},t) = p_0\, e^{-(t-r/c)/t_0} \left(\frac{\sin\theta\,\hat{\boldsymbol{\theta}}}{c^2 t_0^2 r} + (2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \left(\frac{1}{r^3} - \frac{1}{c\, t_0 r^2} \right) \right), \tag{52}$$

$$\mathbf{B}(\mathbf{r},t) = p_0\, e^{-(t-r/c)/t_0} \left(\frac{1}{c^2 t_0^2 r} - \frac{1}{c\, t_0 r^2} \right) \sin\theta\,\hat{\boldsymbol{\phi}}. \tag{53}$$

The radial component of the Poynting vector for $t > r/c$ is,

$$S_r(\mathbf{r},t) = \frac{p_0^2\, e^{-2(t-r/c)/t_0} \sin^2\theta}{4\pi t_0 r^2} \left(\frac{1}{c^3 t_0^3} - \frac{2}{c^2 t_0^2 r} + \frac{2}{c\, t_0 r^2} - \frac{1}{r^3} \right). \tag{54}$$

The sign of S_r is independent of time, is positive for large r and negative for small r, and S_r vanishes when,

$$r^3 - 2c\, t_0 r^2 + 2c^2 t_0^2 - c^3 t_0^3 = (r - c\, t_0)\left(r^2 - c\, t_0 r + c^2 t_0^2\right) = 0, \tag{55}$$

i.e., when $r = c\, t_0$.

We are led to say that the time-dependent dipole for $t > 0$ does not emit any "radiation", but rather absorbs all the energy stored at $r < c\,t_0$, which energy is "radiated" by the vacuum back towards the "source" charges. Meanwhile, all of the energy $p_0^2/3c^3t_0^3$ stored at $r > c\,t_0$ is "radiated to infinity". The "source" of this "radiation" is not the physical dipole, but the electromagnetic field energy stored at $r > c\,t_0$. Of course, as seen in sec. 2.5, this stored energy is due to emission of energy by the source dipole at a much earlier time, when it rose from zero to p_0. Hence, we could say that the "radiation to infinity" is ultimately due to the much earlier charge separation that established the static field $\mathbf{E}_0$ that existed before the decay of the dipole back to zero.

This example was first discussed by Mandel [23]. See also [24].

2.5.1 The Moment Rises Exponentially to p_0

It may be of interest to reconsider the example of sec. 2.6 now supposing that the moment rises exponentially from zero at $t = 0$ such that,

$$\mathbf{p}(t) = p_0\,\hat{\mathbf{z}} \begin{cases} 0 & (t < 0), \\ 1 - e^{-t/t_0} & (t > 0). \end{cases} \tag{56}$$

Similar to the discussion at the beginning of sec. 2.5.3, after time $t = r/c$ the field and the stored energy at radius r rise from zero, and at large times they are both constant. Some of the energy emitted by the dipole becomes stored in the electric field, and some is "radiated to infinity". For times $t \gg r/c$ the magnetic field is zero, the electric field is,

$$\mathbf{E}_0(\mathbf{r}) = \frac{3(\mathbf{p}_0 \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}_0}{r^3} = \frac{p_0(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})}{r^3} \qquad (t \gg r/c), \tag{57}$$

and the electric field energy stored outside a sphere of radius $r_0 \gg a$ at times $t \gg r_0/c$ is,

$$U_0 = \int_{r>r_0} \frac{E_0^2}{8\pi}\,d^3\mathbf{x} = 2\pi \int_{r_0}^{\infty} r^2\,dr \int_{-1}^{1} d\cos\theta\; \frac{p_0^2(3\cos^2\theta + 1)}{8\pi r^6} = \frac{p_0^2}{3r_0^3}\,. \tag{58}$$

In more detail, we note that according to an observer at radius r at time $t > r/c$ the retarded magnetic moment are its derivatives are,

$$[\mathbf{p}] = p_0\,(1 - e^{-(t-r/c)/t_0})\,\hat{\mathbf{z}}, \qquad [\dot{\mathbf{p}}] = \frac{p_0}{t_0}\,e^{-(t-r/c)/t_0}\,\hat{\mathbf{z}}, \qquad [\ddot{\mathbf{p}}] = -\frac{p_0}{t_0^2}\,e^{-(t-r/c)/t_0}\,\hat{\mathbf{z}}, \tag{59}$$

so the fields for $t > r/c$ follow from eqs. (7)-(8) as,

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= -p_0\,e^{-(t-r/c)/t_0}\left(\frac{\sin\theta\,\hat{\boldsymbol{\theta}}}{c^2t_0^2r} - \frac{(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})}{c\,t_0r^2}\right) \\ &\quad + \frac{p_0\,(1 - e^{-(t-r/c)/t_0})}{r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}), \end{aligned} \tag{60}$$

$$\mathbf{B}(\mathbf{r},t) = -p_0\,e^{-(t-r/c)/t_0}\left(\frac{1}{c^2t_0^2r} - \frac{1}{c\,t_0r^2}\right)\sin\theta\,\hat{\boldsymbol{\phi}}. \tag{61}$$

The radial component of the Poynting vector for $t > r/c$ is,

$$
\begin{aligned}
S_r(\mathbf{r},t) &= \frac{p_0^2\, e^{-2(t-r/c)/t_0} \sin^2\theta}{4\pi t_0 r^2} \left(\frac{1}{c^3 t_0^3} - \frac{2}{c^2 t_0^2 r} + \frac{1}{c\, t_0 r^2} \right) \\
&\quad + \frac{p_0^2\, e^{-(t-r/c)/t_0}\,(1 - e^{-(t-r/c)/t_0}) \sin^2\theta}{4\pi t_0 r^2} \left(\frac{1}{r^3} - \frac{1}{c\, t_0 r^2} \right),
\end{aligned}
\tag{62}
$$

which is positive for all r. The total energy that crosses a sphere of radius r is,

$$
U_r = 2\pi \int_{-1}^{1} d\cos\theta \int_{r/c}^{\infty} r^2 S_r(\mathbf{r},t)\, dt = \frac{p_0^2}{3} \left(\frac{1}{r^3} + \frac{1}{c^3 t_0^3} - \frac{2}{c^2 t_0^2 r} \right). \tag{63}
$$

Of this, the first term is the energy U_0 of eq. (58) the remains stored in the fields at finite r and large times. Hence, we infer that the "radiation to infinity" at radius r is,

$$
U_{\infty,r} = \frac{p_0^2}{3} \left(\frac{1}{c^3 t_0^3} - \frac{2}{c^2 t_0^2 r} \right). \tag{64}
$$

However, according to eq. (30) the total energy "radiated to infinity" is,

$$
U_\infty = \int \frac{2\,|[\ddot{\mathbf{p}}]|^2}{3c^3}\, dt = \int_0^\infty \frac{2 p_0^2 e^{-2t_{\rm source}/t_0}}{3c^3 t_0^4}\, dt_{\rm source} = \frac{p_0^2}{3c^3 t_0^3}. \tag{65}
$$

Thus, the amount of "radiation to infinity" that crosses a sphere of radius r is less than the total "radiation to infinity". As in the example of secs. 2.5, part of the "radiation to infinity" comes from energy stored in the fields at large r, and which energy did not arrive there via the Poynting vector $\mathbf{S}_{1/r^2}$ associated with "radiation to infinity" at earlier times.

A Appendix: Fields Calculated from the Retarded Potentials

This Appendix follows [9].

We consider a time-dependent point dipole centered at the origin, as defined by,

$$
\mathbf{p}(t) = \lim_{q\to\infty,\ d\to 0,\ qd=p} q(t)\mathbf{d}, \tag{66}
$$

for which the associated electric charge density ρ can be written,

$$
\varrho(\mathbf{r},t) = \lim_{q\to\infty,\ d\to 0,\ qd=p} q(t)[\delta^3(\mathbf{r} - \mathbf{d}/2) - \delta^3(\mathbf{r} - \mathbf{d}/2) = \mathbf{p}(t)\cdot\boldsymbol{\nabla}\delta^3(\mathbf{r}). \tag{67}
$$

The current density $\mathbf{J}$ is related by the equation of continuity,

$$
\boldsymbol{\nabla}\cdot\mathbf{J}(\mathbf{r},t) = -\frac{\partial\varrho(\mathbf{r},t)}{\partial t} = \dot{\mathbf{p}}(t)\cdot\boldsymbol{\nabla}\delta^3(\mathbf{r}) = \boldsymbol{\nabla}\cdot[\dot{\mathbf{p}}(t)\,\delta^3(\mathbf{r})], \tag{68}
$$

so that,

$$
\mathbf{J}(\mathbf{r},t) = \dot{\mathbf{p}}(t)\,\delta^3(\mathbf{r}). \tag{69}
$$

The retarded scalar potential V is given (in Gaussian units) by,

$$\begin{aligned} V(\mathbf{r},t) &= \int \frac{\varrho(\mathbf{r}', t' = t - |\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}\, d^3\mathbf{r}' = \int \frac{\mathbf{p}(t')\cdot\boldsymbol{\nabla}\delta^3(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3\mathbf{r}' \\ &= -\int \delta^3(\mathbf{r}')\boldsymbol{\nabla}\cdot\frac{\mathbf{p}(t')}{|\mathbf{r}-\mathbf{r}'|}\, d^3\mathbf{r}' = -\boldsymbol{\nabla}\cdot\frac{\mathbf{p}(t' = t - r/c)}{r} \\ &= \frac{[\mathbf{p}]\cdot\mathbf{r}}{r^3} + \frac{[\dot{\mathbf{p}}]\cdot\mathbf{r}}{c\,r^2}, \end{aligned} \tag{70}$$

where we write a retarded quantity $f(t - r/c)$ as $[f]$, and note that $\boldsymbol{\nabla} r = \mathbf{r}/r$ and,

$$\boldsymbol{\nabla}\cdot\mathbf{p}(t - r/c) = -\frac{[\dot{\mathbf{p}}]}{c}\cdot\boldsymbol{\nabla} r = -\frac{[\dot{\mathbf{p}}]\cdot\mathbf{r}}{c\,r}. \tag{71}$$

Similarly, the retarded vector potential $\mathbf{A}$ is given by,

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{r}', t' = t - |\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}\, d^3\mathbf{r}' = \int \frac{\dot{\mathbf{p}}(t')\,\delta^3(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3\mathbf{r}' = \frac{[\dot{\mathbf{p}}]}{c\,r}. \tag{72}$$

The electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ are obtained from the retarded potentials according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \tag{73}$$

noting that $\boldsymbol{\nabla}\times\mathbf{r} = 0$,

$$\boldsymbol{\nabla}\times\mathbf{p}(t - r/c) = -\frac{\boldsymbol{\nabla} r}{c}\times[\dot{\mathbf{p}}] = -\frac{\mathbf{r}}{c\,r}\times[\dot{\mathbf{p}}],. \tag{74}$$

and,

$$\begin{aligned} \boldsymbol{\nabla}([\mathbf{p}]\cdot\mathbf{r}) &= ([\mathbf{p}]\cdot\boldsymbol{\nabla})\mathbf{r} + (\mathbf{r}\cdot\boldsymbol{\nabla})[\mathbf{p}] + [\mathbf{p}]\times(\boldsymbol{\nabla}\times\mathbf{r}) + [\mathbf{r}\times(\boldsymbol{\nabla}\times\mathbf{p}]) \\ &= [\mathbf{p}] - [\dot{\mathbf{p}}]\frac{r}{c} + 0 + [\dot{\mathbf{p}}]\frac{r}{c} - \frac{([\dot{\mathbf{p}}]\cdot\mathbf{r})\mathbf{r}}{c\,r} = [\mathbf{p}] - \frac{([\dot{\mathbf{p}}]\cdot\mathbf{r})\mathbf{r}}{c\,r} \end{aligned} \tag{75}$$

Thus,

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla}\frac{[\mathbf{p}]\cdot\mathbf{r}}{r^3} - \boldsymbol{\nabla}\frac{[\dot{\mathbf{p}}]\cdot\mathbf{r}}{c\,r^2} - \frac{1}{c}\frac{\partial}{\partial t}\frac{[\dot{\mathbf{p}}]}{c\,r} \\ &= -\frac{1}{r^3}\boldsymbol{\nabla}([\mathbf{p}]\cdot\mathbf{r}) - ([\mathbf{p}]\cdot\mathbf{r})\boldsymbol{\nabla}\frac{1}{r^3} - \frac{1}{c\,r^2}\boldsymbol{\nabla}([\dot{\mathbf{p}}]\cdot\mathbf{r}) - ([\dot{\mathbf{p}}]\cdot\mathbf{r})\boldsymbol{\nabla}\frac{1}{c\,r^2} - \frac{[\ddot{\mathbf{p}}]}{c^2 r} \\ &= -\frac{[\mathbf{p}]}{r^3} + \frac{([\dot{\mathbf{p}}]\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{c\,r^2} + 3\frac{([\mathbf{p}]\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{r^3} - \frac{[\dot{\mathbf{p}}]}{c\,r^2} + \frac{([\ddot{\mathbf{p}}]\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{c^2 r} + 2\frac{([\dot{\mathbf{p}}]\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{c\,r^2} - \frac{[\ddot{\mathbf{p}}]}{c^2 r} \\ &= \frac{([\ddot{\mathbf{p}}]\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{c^2 r} + \frac{3([\dot{\mathbf{p}}]\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - [\dot{\mathbf{p}}]}{c\,r^2} + \frac{3([\mathbf{p}]\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - [\mathbf{p}]}{r^3}, \end{aligned} \tag{76}$$

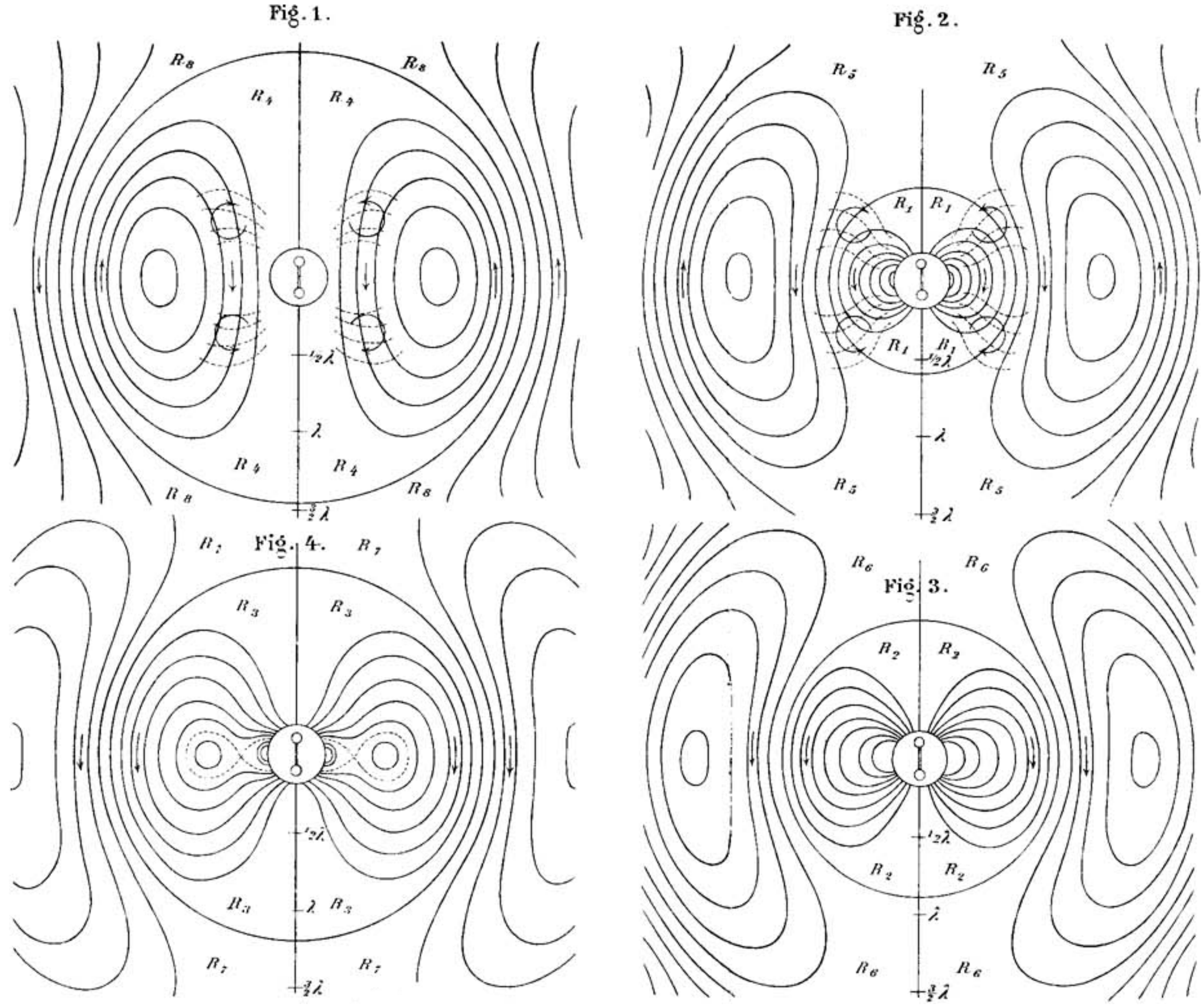
and,

$$\mathbf{B} = \boldsymbol{\nabla}\times\frac{[\dot{\mathbf{p}}]}{c\,r} = \frac{1}{c\,r}\boldsymbol{\nabla}\times[\dot{\mathbf{p}}] + \left(\boldsymbol{\nabla}\frac{1}{c\,r}\right)\times[\dot{\mathbf{p}}] = -\frac{\hat{\mathbf{r}}}{c^2 r}\times[\ddot{\mathbf{p}}] - \frac{\hat{\mathbf{r}}}{c\,r^2}\times[\dot{\mathbf{p}}], \tag{77}$$

in agreement with eqs. (7)-(8).

A.1 Fields Near the Origin (July 15, 2020)[4]

The fields (3) of an oscillating electric dipole were first deduced by Hertz [25], who illustrated a half cycle of their history in the near zone in the figures below.



In Fig. 1 the dipole moment $\mathbf{p}$ is instantaneously zero.

In Fig. 3 the dipole moment has its maximum value, with the + charge on top, and the − charge below.

In Figs. 2 and 4 the dipole moment has 1/2 its maximum value, which grows between Figs. 2 and 3, but shrinks between Figs. 3 and 4.

The next four figures in the sequence, to complete one cycle, would look just like Figs. 1, 2, 3 and 4, but with the directions of the field lines reversed.

These figures indicate how Hertz understood that the "point" dipole resides inside a very small sphere, and that his solution for the fields applies only outside this sphere, where there are no sources and the fields obey the source-free wave equation $\nabla^2\mathbf{E},\mathbf{B} = \partial^2\mathbf{E},\mathbf{B}/\partial(ct)^2$.

Turning to the issue of the fields near the origin, we first consider the magnetic field (77), which is purely azimuthal with respect to the vector $\mathbf{p}$, and hence vanishes at the origin. We

[4] This section was suggested by Vladimir Onoochin.

signify this by rewriting eq. (77) as,

$$\mathbf{B} = \mathbf{0} - \frac{\hat{\mathbf{r}}}{c^2 r} \times [\ddot{\mathbf{p}}] - \frac{\hat{\mathbf{r}}}{c\, r^2} \times [\dot{\mathbf{p}}], \tag{78}$$

where the terms in $1/r$ and $1/r^2$ are defined only for $r > 0$, and $\mathbf{0}$ is a field of zero-lengths vectors.

The field (78) should obey the wave equation (obtained from Maxwell's equations),

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\frac{4\pi}{c} \boldsymbol{\nabla} \times \mathbf{J}. \tag{79}$$

From eq. (69), we have that $\mathbf{J} = \dot{\mathbf{p}}\, \delta^3(\mathbf{r})$. Integrating this over a small sphere,

$$\int_{r<R} \boldsymbol{\nabla} \times \mathbf{J}\, d\mathrm{Vol} = \int_{r=R} \mathbf{J} \cdot d\mathbf{Area} = \int_{r=R} \dot{\mathbf{p}}\, \delta^3(\mathbf{r}) \cdot d\mathbf{Area} = 0, \tag{80}$$

from which we infer that $\boldsymbol{\nabla} \times \mathbf{J} = 0$ everywhere. That is, the magnetic field must satisfy the wave equation $\nabla^2 \mathbf{B} = \partial^2 \mathbf{B}/\partial(ct)^2$ everywhere. As we already know that this is satisfied for all nonzero $\mathbf{R}$, we simply define it to be satisfied at the or-gin as well.

In the electric field (75), the first term is transverse to $\mathbf{r}$ and cannot contribute at the origin. Near the origin, the second, $1/r^2$, is negligible compared to the third, $1/r^3$ term. Also, retardation is negligible near the origin, so the third term is essentially the instantaneous static electric-dipole field. As, discussed in sec. 4.3 of [7], the static dipole field at the origin can be represented by the term $-4\pi \mathbf{p}\, \delta^3(\mathbf{r})/3$. Hence, the electric field including the term at the origin can be written as,

$$\mathbf{E} = \frac{([\ddot{\mathbf{p}}] \times \hat{\mathbf{r}}) \times \hat{\mathbf{r}}}{c^2 r} + \frac{3([\dot{\mathbf{p}}] \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}} - [\dot{\mathbf{p}}]}{c\, r^2} + \frac{3([\mathbf{p}] \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}} - [\mathbf{p}]}{r^3} - \frac{4\pi}{3} \mathbf{p}\, \delta^3(\mathbf{r}), \tag{81}$$

with the convention that the first three terms are defined only for nonzero $\mathbf{r}$.

The field (81) should obey the wave equation (obtained from Maxwell's equations),

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 4\pi \boldsymbol{\nabla} \varrho + \frac{4\pi}{c^2} \frac{\partial \mathbf{J}}{\partial t} = 4\pi \boldsymbol{\nabla}(\mathbf{p} \cdot \boldsymbol{\nabla} \delta^3(\mathbf{r})) + \frac{4\pi}{c^2} \ddot{\mathbf{p}}\, \delta^3(\mathbf{r}), \tag{82}$$

recalling eqs. (67) and (69).

At the origin, where only the last term of eq. (81) contributes, we have that

$$-\frac{1}{c^2} \frac{\partial^2 \mathbf{E}(r=0)}{\partial t^2} = \frac{4\pi}{3c^2} \ddot{\mathbf{p}}\, \delta^3(\mathbf{r}=0). \tag{83}$$

Can we say that,

$$\nabla^2 \mathbf{E}(r=0) = 4\pi \boldsymbol{\nabla}(\mathbf{p} \cdot \boldsymbol{\nabla} \delta^3(\mathbf{r}=0)) + \frac{8\pi}{3c^2} \ddot{\mathbf{p}}\, \delta^3(\mathbf{r}=0), \tag{84}$$

such that eq. (82) is satisfied everywhere?

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