Power Consumption by a Pulsed Copper Magnet

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1 Problem

A resistive copper magnet is designed to generate a magnetic field of specified strength over a specified volume, but this magnetic field is needed only at regular intervals time T apart. Discuss how pulsed operation of the magnet could reduce its average power consumption, at the expense of greater peak power (and voltage) that must be delivered by the power supply.

Comment on the case that the magnet time constant $\tau = L/R$ is large compared to T.

2 Solution

This problem was suggested by John Seeman. The solution follows a suggestion by Bob Palmer, and elaborates on a discussion by Bob Weggel [1].

2.1 DC Operation

The requirement of a specified magnetic field over a specified volume roughly constrains the stored energy,

$$U = \int \frac{B^2}{2\mu_0}\, d\text{Vol}. \tag{1}$$

If the copper coils of the magnet occupy a specified volume (surrounding the "good" field region), this fixes the current density $\mathbf{J}$ required to generate the field. For DC operation of the magnet its power consumption is then given by

$$P = \int_{\text{coils}} \frac{J^2}{\sigma}\, d\text{Vol}, \tag{2}$$

where σ is the electrical conductivity of copper. The DC power consumption is independent of the details of the winding of the coil, although its latter determines the inductance L, its resistance R and the DC current I.

2.2 Invariance of the Magnet Time Constant

If we consider transient operation of the magnet, we are led to an awareness of the time constant

$$\tau = \frac{L}{R}\,. \tag{3}$$

The stored energy U and the DC power consumption P are related to the DC current $I_{\rm DC}$ by

$$U_{\rm DC} = \frac{L I_{\rm DC}^2}{2}, \qquad \text{and} \qquad P_{\rm DC} = I_{\rm DC}^2 R. \tag{4}$$

Hence, the time constant can also be written as

$$\tau = \frac{2U_{\rm DC}}{P_{\rm DC}}, \tag{5}$$

which is independent of the details of the winding.

2.3 Pulsed Operation

To deliver the specified magnetic field only at times space by the interval T, including $t = 0$, we consider a current waveform

$$I(t) = \sum_{n=0}^{\infty} I_n \cos n\omega t = Re \sum_{n=0}^{\infty} I_n\, e^{ni\omega t}, \tag{6}$$

where $\omega = 2\pi/T$ and the Fourier coefficients I_n are related to the DC current $I_{\rm DC}$ by

$$I_{\rm DC} = \sum_{n=0}^{\infty} I_n. \tag{7}$$

The power supply must deliver voltage given by

$$\begin{aligned} V &= L\dot{I} + IR = Re \sum_{n=0}^{\infty} I_n (ni\omega L + R)\, e^{ni\omega t} = \sum_{n=0}^{\infty} I_n (R \cos n\omega t - n\omega L \sin n\omega t) \\ &= R \sum_{n=0}^{\infty} I_n (\cos n\omega t - n\omega\tau \sin n\omega t). \end{aligned} \tag{8}$$

The peak voltage is the maximum of $I_0 R$ and $\omega\tau R \sum_{n=1}^{\infty} n I_n$. When $T \ll \tau$, such that $\omega\tau \gg 1$, the peak voltage will be large compared to $I_{\rm DC} R = V_{\rm DC}$, and the power supply may be difficult to design.

The time-average power delivered by the supply is

$$\begin{aligned} \langle P \rangle &= \frac{1}{T} \int_0^T V I\, dt = \frac{1}{T} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} I_m I_n \left(R \int_0^T \cos m\omega t \cos n\omega t\, dt - n\omega L \int_0^T \cos m\omega t \sin n\omega t\, dt \right) \\ &= \left(I_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} I_n^2 \right) R = \frac{1}{2} \left(I_0^2 + \sum_{n=0}^{\infty} I_n^2 \right) R. \end{aligned} \tag{9}$$

2.3.1 Single Harmonic

If we operate at only harmonic $n > 0$, then we must have $I_n = I_{\rm DC}$, which leads to $\langle P_n \rangle = P_{\rm DC}/2$.

The power-supply voltage is

$$V = I_{\rm DC}(R \cos n\omega t - n\omega L \sin n\omega t) = I_{\rm DC} R (\cos n\omega t - n\omega\tau \sin n\omega t), \tag{10}$$

The peak voltage is $I_{\rm DC}R = V_{\rm DC}$ times the maximum of 1 and $n\omega t$. When $T \ll \tau$, such that $\omega\tau \gg 1$, the peak voltage is

$$V_{\rm max} = n\omega\tau I_{\rm DC}R = n\omega\tau V_{\rm DC} \gg V_{\rm DC} \qquad (\omega\tau \gg 1). \tag{11}$$

In principle, we could use an alternative winding that keeps U, P and τ the same, but has different values I', L' and R' for I, L and R, subject to

$$\frac{L'}{L} = \frac{R'}{R} = \sqrt{\frac{I}{I'}}\,. \tag{12}$$

Then the peak voltage would be the maximum of $V_{\rm DC}$ and

$$V'_{\rm max} = n\omega\tau I'R' = n\omega\tau IR\sqrt{\frac{I'}{I}} = n\omega\tau\sqrt{\frac{I'}{I}}V_{\rm DC}. \tag{13}$$

By choosing $I' = I/(n\omega\tau)^2$ we could achieve a peak voltage/power equal to $V_{\rm DC}$ and an average power one half the DC value.

This is a huge effort for very little reward; and it may not be practical to wind the magnet with a very large number of turns so that I' is small and L' and R' are large.

2.3.2 Geometric Series

An ideal current waveform would be $I(t) = I_{\rm DC}\,\delta(t)$, for which the Fourier coefficients are $I_0 = I_{\rm DC}$ and $I_n = 2I_{\rm DC}$ for $n \geq 1$. However, the validity of the series expansion depends on the quality of the very high harmonics, which is doubtful in practice. So, we seek a series in which the Fourier coefficients are nearly equal, but die out slowly at large n, as a reasonable approximation to a δ-function.

A simple infinite series of coefficients I_n that sum to $I_{\rm DC}$ is the geometric series

$$I_n = \frac{a-1}{a^{n+1}}I_{\rm DC}, \tag{14}$$

for $a > 1$. For a slightly larger than 1 the current waveform will be close to a δ-function. The average power follows from eq. (9) as

$$\langle P_{\rm geometric}\rangle = \frac{(a-1)^2 I_{\rm DC}^2 R}{2a^2}\left(1 + \sum_{n=0}^{\infty}\frac{1}{a^{2n}}\right) = \frac{(a-1)(2-1/a^2)}{2(a+1)}P_{\rm DC} \approx \frac{a-1}{4}P_{\rm DC}, \tag{15}$$

where the approximation holds for a slightly larger than 1. The peak voltage when $\omega\tau \gg 1$ is

$$V_{\rm max} = \omega\tau R\sum_{n=1}^{\infty} nI_n = \frac{a-1}{a}\omega\tau V_{\rm DC}\sum_{n=1}^{\infty}\frac{n}{a^n} = \frac{\omega t}{a+1}V_{\rm DC} \qquad (\omega\tau \gg 1). \tag{16}$$

To have a substantial reduction in power with a slightly larger than 1, the peak voltage/power will be essentially $\omega\tau V_{\rm DC}$.

For the example of [1] where $\tau = 0.71$ s, $\omega = 2\pi(1/15) = 94.2$ s^{-1}, and $V_{\rm DC} = 913$ V, $\omega\tau V_{\rm DC} \approx 61$ kV. Whether the power supply, and the magnet, can deal with this high voltage remains to be considered.

When a is only slightly larger than 1 the geometric series converges slowly, and in practice a large number of terms would need to be kept. For low ω, as in [1], even rather high harmonics have low frequency compared to the GHz scale that is readily achievable with contemporary electronic circuits, so the geometric-series waveform could be well realized in the power supply.

References

[1] R. Weggel, *Minimum-Heating IDS120H Resistive Magnet Pulsed at 15 Hz* (July 15, 2011), http://physics.princeton.edu/~mcdonald/mumu/target/weggel/Pulse_min-heat.pdf