Fields inside a Sphere and Shell of Uniform Polarization

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1 Problem

Deduce the fields of a hollow shell (and also of a "solid" sphere) of radius a that supports a uniform surface (or volume) polarization density, either electric or magnetic.

2 Solution

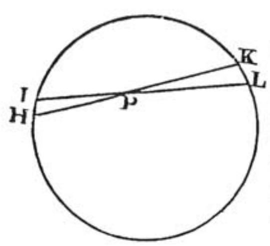
2.1 Newton

We recall the well-known geometric argument due to Newton, p. 218 of [1], that the gravitational force (electric field) is zero in the interior of a spherical shell that has a uniform surface mass (charge) density.

Of the attractive forces of sphærical bodies.

PROPOSITION LXX. THEOREM XXX.

If to every point of a sphærical surface there tend equal centripetal forces decreasing in the duplicate ratio of the distances from those points; I say, that a corpuscle placed within that superficies will not be attracted by those forces any way.



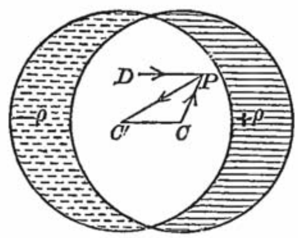
Let HIKL, be that sphærical superficies, and P a corpuscle placed within. Through P let there be drawn to this superficies to two lines HK, IL, intercepting very small arcs HI, KL; and because (by Cor. 3, Lem. VII) the triangles HPI, LPK are alike, those arcs will be proportional to the distances HP LP; and any particles at HI and KL of the sphærical superficies, terminated by right lines passing through P, will be in the duplicate ratio of those distances. Therefore the forces of these particles exerted upon the body P are equal between themselves. For the forces are as the particles directly, and the squares of the distances inversely. And these two ratios compose the ratio of equality. The attractions therefore, being made equally towards contrary parts, destroy each other. And by a like reasoning all the attractions through the whole sphærical superficies are destroyed by contrary attractions. Therefore the body P will not be any way impelled by those attractions. Q.E.D.

In the following Proposition LXXI, Newton demonstrated that a point mass at distance $r > a$ from the center of a shell of mass M is attracted to that center with a force proportional M/r^2.

2.2 W. Thomson (Lord Kelvin)

We next recall an argument due to W. Thomson on pp. 470-471 of [2] (see also p. 362 where aspects of the argument were attributed to Poisson), which is reproduced below.

> To find the resultant due to one such distribution of matter on a spherical surface, imagine first a solid material globe of uniform volume-density ρ throughout. By Newton's theorems for the attraction of a uniform spherical mass, acting according to his law of the inverse square of the distance, the resultant force at any point within the substance will be towards the centre, and equal to $\frac{4\pi\rho}{3}$ multiplied by the distance of the attracted point from the centre of the globe. Consider now two equal globes, one of uni-



> form positive matter and the other of uniform negative matter of the same density, the former repelling and the latter attracting a unit of positive matter (as in the electric and magnetic applications of the Newtonian law). Let them be placed with their centres C and C', at any distance apart less than the sum of their radii, and first imagine their materials to co-exist in the space common to the two spherical volumes, each acting as if the other were away. The resultant force at any point P within this space will be found by compounding a force equal to $\frac{4\pi\rho}{3}CP$ with a force $\frac{4\pi\rho}{3}C'P$, in the direction from P towards C', and therefore, according to the parallelogram of forces, will be in the direction PD parallel to CC', and will be equal to $\frac{4\pi\rho}{3}CC'$. This (as the positive and negative matters in the space common to the two spheres neutralize one another) is therefore the resultant force at P, due to uniform distribution of positive and negative matter in the two meniscuses formed by the non-coincident portions of the two spheres. Now let CC' become infinitely small, and ρ infinitely large, and denote by i the product $\rho CC'$, which we may suppose to have any value we please. The two meniscuses become a continuous superficial distribution of matter over a single spherical surface, having for surface-density $i\cos\theta$, at any point where the inclination of the normal to the diameter through CC' is θ. The resultant force is parallel to this diameter and of constant value equal to $\frac{4\pi i}{3}$ throughout the entire spherical space.

When applying this argument to electricity and magnetism, where like charge repel (in contrast to the gravitational attraction between (like) masses), there is a reversal of sign as to the force/field inside the sphere. Also, we note that Thomson's (vector) symbol $\mathbf{i}$ (for intensity) corresponds to the volume density of dipoles, assumed to be formed of equal and opposite charges. We transcribe this symbol as $\mathbf{P}_e$ for the case of electric dipoles due to electric charges, and $\mathbf{M}_m$ for the case of magnetic dipoles due to hypothetical magnetic charges (*i.e.*, monopoles, as was assumed by Thomson, who followed Poisson on this). Then, the field inside the sphere (of radius a) can be written (in Gaussian units) as[1,2]

$$\mathbf{E} = -\frac{4\pi \mathbf{P}_e}{3}, \qquad \mathbf{D} = \mathbf{E} + 4\pi \mathbf{P} = \frac{8\pi \mathbf{P}_e}{3}, \qquad \mathbf{B} = -\frac{4\pi \mathbf{M}_m}{3}, \qquad \mathbf{H} = \mathbf{B} - 4\pi \mathbf{M} = -\frac{16\pi \mathbf{M}_m}{3}. \tag{2}$$

A corollary to Thomson's argument (not immediately stated by him) is that the fields exterior to the sphere are as if the positive and negative charges of the dipoles were concentrated at the centers of their respective spheres. That is, the exterior fields are those associated with dipoles $\mathbf{p}_e = 4\pi a^3 \mathbf{P}_e/3$ or $\mathbf{m}_m = 4\pi a^3 \mathbf{M}_m/3$, namely

$$\mathbf{D} = \mathbf{E} = \frac{3(\mathbf{p}_e \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}_e}{r^3}, \qquad \mathbf{B} = \mathbf{H} = \frac{3(\mathbf{m}_m \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}_m}{r^3} \qquad (r > a). \tag{3}$$

However, it can be that magnetic dipoles are due to electric currents, and electric dipoles are due to (hypothetical) magnetic currents. The fields inside uniform spheres of such polarization are not described by the above argument of Thomson, so we seek other insights.

2.3 The Field inside a Uniformly Polarized Spherical Shell is Zero

In the spirit of Thomson's argument, a uniformly polarized spherical shell can be thought of as the sum of two shells of opposite, uniform charge density, slightly offset. Then, since the field inside each of these uniformly charged shells is zero, it is also zero inside their common interior volume, *i.e.*, inside the uniformly polarized spherical shell.

Further, this result holds for polarization due to currents as well as to charges (as assumed in the previous paragraph) in that the field inside the polarized shell is due entirely to the fields external to the dipoles on the shell, and these external fields are the same for dipoles based on pairs of charges and on current loops.

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[1]The magnetic fields called **B** and **H** by Maxwell in Arts. 398-399 of [6] were first distinguished by Thomson in 1871, p. 397 of [2]. Thomson considered that his argument sketched above applied to **H** rather than to **B**. We now justify this view by considering the magnetic charges to be "fictitious" equivalent sources of magnetic fields actually due to (Ampèrian) electric-current loops. Then, we could rewrite the third and fourth relations of eq. (2) as

$$\mathbf{H} = -\frac{4\pi \mathbf{M}_e}{3}, \qquad \mathbf{B} = \mathbf{H} + 4\pi \mathbf{M} = \frac{8\pi \mathbf{M}_e}{3}, \tag{1}$$

which we deduce by a different argument in sec. 2.5 below.

[2]Neither Thomson nor Maxwell enunciated a concept of the polarization density **P** of electric dipoles, and only regarded the relation between **D** and **E** as $\mathbf{D} = \epsilon \mathbf{E}$, where ϵ is now called the (relative) dielectric constant and/or the (relative) permittivity. See Art. 111 of [7] for Maxwell's use of the term polarization.

In 1885, Heaviside introduced the concept of an electret as the electrical analog of a permanent magnet [8], and proposed that the electrical analog of magnetization (density) be called electrization. He did not propose a symbol for this, nor did he write an equation such as $\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}$.

The density of electric dipoles was called the polarization by Lorentz (1892) in sec. 102, p. 465 of [9], and assigned the symbol **M**.

Larmor (1895), p. 738 of [10], introduced the vector (f', g', h') for what is now written as the polarization density **P**, and related it to the electric field $\mathbf{E} = (P, Q, R)$ as $(f', g', h') = (K - 1)(P, Q, R)/4\pi$, *i.e.*, $\mathbf{P} = (\epsilon - 1)\mathbf{E}/4\pi = (\mathbf{D} - \mathbf{E})/4\pi$. Larmor's notation was mentioned briefly on p. 91 of [11] (1898).

The symbol **M** for dielectric polarization was changed to **P** by Lorentz on p. 263 of [12] (1902), and a relation equivalent to $\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}$ was given in eq. (22), p. 265. See also p. 224, and eq. (147), p. 240 of [13] (1903), which latter subsequently appeared as eq. (142), p. 155 of the textbook [14] (1904) by Abraham.

2.4 The Field at the Center of a Uniformly Polarized Sphere

In the continuum approximation, a uniformly polarized sphere is equivalent to a set of nested spherical shells, plus a tiny polarized sphere about the center of the larger sphere. The field at the center of the larger sphere due to each of the nested spherical shells is zero, according to the argument of sec. 2.3 above. Hence, the field at the center of the larger sphere is the same as the field at the center of the tiny polarized sphere about the center.

We recall that the fields at the center of a "point" dipole $\mathbf{p}_e$ or $\mathbf{m}_m$ due to equal and opposite charges, and that of $\mathbf{m}_e$ or $\mathbf{p}_m$ due to current loops, are given by[3]

$$\mathbf{E} = -\frac{4\pi \mathbf{p}_e}{3}\delta^3\mathbf{r}), \qquad \mathbf{B} = -\frac{4\pi \mathbf{m}_m}{3}\delta^3\mathbf{r}), \qquad \mathbf{E} = \frac{8\pi \mathbf{p}_m}{3}\delta^3\mathbf{r}), \qquad \mathbf{B} = \frac{8\pi \mathbf{m}_e}{3}\delta^3\mathbf{r}). \tag{4}$$

The moments in the tiny sphere of volume dVol about the center of the larger sphere are $\mathbf{P}_e\, d$Vol, *etc.*, so we infer that the fields at the center of the larger, uniformly polarized spheres are

$$\mathbf{E} = -\frac{4\pi \mathbf{P}_e}{3}, \qquad \mathbf{B} = -\frac{4\pi \mathbf{M}_m}{3}, \qquad \mathbf{E} = \frac{8\pi \mathbf{P}_m}{3}, \qquad \mathbf{B} = \frac{8\pi \mathbf{M}_e}{3}. \tag{5}$$

2.5 The Field inside a Uniformly Polarized Sphere due to Current Loops

The first and second relations of eq. (5) agree with the analysis of sec. 2.2 for a sphere of uniform polarization due to charges, where the field was found to be uniform throughout the sphere. Hence, we extrapolate that fields inside uniformly polarized spheres due to dipoles based on current loops are also uniform throughout the sphere, but with values given by the third and fourth relations of eq. (5).[4]

$$\mathbf{E} = \frac{8\pi \mathbf{P}_m}{3}, \qquad \mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \frac{16\pi \mathbf{P}_m}{3}, \qquad \mathbf{B} = \frac{8\pi \mathbf{M}_e}{3}, \qquad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M} = -\frac{4\pi \mathbf{M}_e}{3}. \tag{6}$$

Now, the field at an arbitrary point $\mathbf{r}$ with a uniformly polarized sphere can be regarded as the sum of the fields due to a tiny sphere about that point plus the field due to the rest of the sphere. Since the fields at the center of a tiny sphere about $\mathbf{r}$ are given by eq. (5), which are also the total fields inside the larger sphere, we infer that the fields at $\mathbf{r}$ due to the rest of the sphere are zero.[5]

[3]See, for example, eq. (5.64) of [4] or sec. III of [5].

[4]This agrees with analyses as in sec. 5.10 of [4], based on a scalar potential for the fields $\mathbf{E}$ or $\mathbf{H}$.

[5]This discussion also relates to the issue of the field inside a small spherical cavity within a polarized medium, where if the cavity were filled the field would be $\mathbf{E}$ or $\mathbf{B}$. The field inside the cavity would then be $\mathbf{E} - \mathbf{E}_{\text{small sphere}}$ or $\mathbf{B} - \mathbf{B}_{\text{small sphere}}$, where the field inside the small sphere is given by the appropriate form of eq. (5).

Such a result was mentioned by W. Thomson on p. 258 of [3] (p. 362 of [2]), who attributed it to Poisson, but a demonstration by Thomson of a key step, eq. (2) above, was only given in 1871, as discussed in sec. 2.2 above. The result was mentioned in the footnote by J.J. Thomson to Art. 400 of the 1892 edition of Maxwell's *Treatise* [6], where the polarization was assumed to be due to charges. See also sec. 4.5 of [4].

In case the medium is a sphere of uniform polarization, the field inside a small cavity would be zero.

2.6 Fields due to Polarization on a Cubic Lattice

The preceeding arguments have all assumed that the polarization densities (like charge and current densities in Maxwellian electrodynamics) are continuous.

A more contemporary view is that the polarization is due to small entities, electrons and/or nuclei, and the Maxwellian description is a macroscopic average over the microscopic fields of these entities.

For example, the polarization might be due to a cubic lattice of two types of entities, like Na and Cl in salt, in which only one type of the entities is polarized, either due to charges or to current loops. The field at the center of one of these polarized entities is described by the appropriate relation in eq. (5), while the field at the center of any of the unpolarized entities would seem to be zero.[6]

In the macroscopic approximation, we can define a (macroscopic) uniform polarization density $\mathbf{P} = \mathbf{p}/d^3$ or $\mathbf{M} = \mathbf{m}/d^3$, where d is the edge length of a unit cell of the lattice of dipoles $\mathbf{p}$ or $\mathbf{m}$, such that the macroscopic fields $\mathbf{E} = \mathbf{D} - 4\pi\mathbf{P}$ and $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$, which are continuous by definition, obey the appropriate form of eq. (5),

The microscopic field in between lattice sites, *i.e.*, between centers of the various entities, varies between zero and the very large value at the centers of the dipoles $\mathbf{p}$ or $\mathbf{m}$.

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[6] A closely related result is that if one of the polarized entities were removed from the lattice, the field due to the remaining polarized entities would be zero at the center of the vacant site, as deduced by Lorentz on p. 306 (Note 55) of [15]. See also sec. 4.5 of [4].

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