

Ph 406: Elementary Particle Physics

Problem Set 3

K.T. McDonald
kirkmed@princeton.edu
Princeton University

Due Monday, October 6, 2014 (updated September 4, 2016)

1. Deduce the nonrelativistic form factors,

$$F(q^2) = \int \rho(r) e^{i\mathbf{q}\cdot\mathbf{r}} d^3\mathbf{r}, \quad (1)$$

for the spherically symmetric charge densities with characteristic radius R ,

$$\rho(r) = \begin{cases} 3Q/4\pi R^3 & (r < R), \\ 0 & (r > R), \end{cases} \quad (2)$$

$$\rho(r) = \frac{Q}{4\pi R^2} \delta(r - R), \quad (3)$$

and

$$\rho(r) = \frac{Q}{2\pi\sqrt{2\pi}R^3} e^{-r^2/2R^2}, \quad (4)$$

all of which have total charge Q . Expand these form factors to order $(qR)^2$.

A neutral particle might have charge distributions ρ_+ and ρ_- with the above forms, but with different values of the characteristic radii R_+ and R_- .

The data are often fit to the form,¹

$$F_n(q^2) = \frac{Q}{[1 + (qR)^2]^n}, \quad (5)$$

with $n = 2$. What are the corresponding forms of the charge distributions $\rho_n(r)$ for $n = 1, 2$ and 3 ?

2. Arbitrary 2×2 Unitary Matrices and Pauli Spin Matrices

This problem concerns operators that act on 2-component spinors. Such operators can be expressed as 2×2 matrices. Operators that preserve the normalization of a state are called **unitary**.

Two of the simplest unitary operators on 2-component spinors are the identity matrix $\mathbf{I}_2 = \mathbf{I}$, and the spin-flip operator \mathbf{X} (called the NOT operator in quantum computation),

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (6)$$

¹For a review of nucleon form factors, see C.F. Perdrisat *et al.*, *Nucleon electromagnetic form factors*, Prog. Part. Nucl. Phys. **59**, 694 (2007),

http://physics.princeton.edu/~mcdonald/examples/EP/perdrisat_ppnp_59_694_07.pdf.

An arbitrary 2×2 unitary matrix \mathbf{U} can be written as

$$\mathbf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (7)$$

where a, b, c and d are complex numbers such that $\mathbf{U}\mathbf{U}^\dagger = \mathbf{I}$. The decomposition (7) is somewhat trivial. Express the general unitary matrix \mathbf{U} as the sum of four unitary matrices, times complex coefficients, of which two are the classical unitary matrices \mathbf{I} and \mathbf{X} given above. Denote the “partner” of \mathbf{I} by \mathbf{Z} and the “partner” of \mathbf{X} by \mathbf{Y} such that

$$\mathbf{X}\mathbf{Y} = i\mathbf{Z}, \quad \mathbf{Y}\mathbf{Z} = i\mathbf{X}, \quad \mathbf{Z}\mathbf{X} = i\mathbf{Y}. \quad (8)$$

You have, of course, rediscovered the so-called Pauli spin matrices,^{2,3}

$$\sigma_x (= \sigma_1) = \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y (= \sigma_2) = \mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z (= \sigma_3) = \mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (9)$$

As usual, we define the Pauli “vector” $\boldsymbol{\sigma}$ as the triplet of matrices

$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z). \quad (10)$$

Show that for ordinary 3-vectors \mathbf{a} and \mathbf{b} ,

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{I} + i \boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{b}. \quad (11)$$

With this, show that a general 2×2 unitary matrix can be written as

$$\mathbf{U} = e^{i\delta} \left(\cos \frac{\theta}{2} \mathbf{I} + i \sin \frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma} \right) = e^{i\delta} e^{i\frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma}}, \quad (12)$$

where δ and θ are real numbers and $\hat{\mathbf{u}}$ is a real unit vector.⁴ By the exponential $e^{\mathbf{O}}$ of an operator \mathbf{O} we mean the Taylor series $\sum_n \mathbf{O}^n / n!$ where $\mathbf{O}^0 = \mathbf{I}$.

What is the determinant of the matrix representation of \mathbf{U} ? The subset of 2×2 unitary matrices with unit determinant is called the **special unitary group** $\text{SU}(2)$. What is the version of eq. (12) that describes 2×2 special unitary operators?

You may wish to convince yourself of a factoid related to eq. (12), namely that if \mathbf{A} is a square matrix of any order such that $\mathbf{A}^2 = \mathbf{I}$, then $e^{i\theta\mathbf{A}} = \cos \theta \mathbf{I} + i \sin \theta \mathbf{A}$, provided that θ is a real number. It follows that \mathbf{A} can also be written in the exponential form

$$\mathbf{A} = e^{i\pi/2} e^{-i\frac{\pi}{2}\mathbf{A}} = e^{-i\pi/2} e^{i\frac{\pi}{2}\mathbf{A}}. \quad (13)$$

²W. Pauli, *Zur Quantenmechanik des magnetischen Elektrons*, Z. Phys. **43**, 601 (1927), http://physics.princeton.edu/~mcdonald/examples/QM/pauli_zp_43_601_27.pdf.

³The Pauli spin matrices (and the unit matrix \mathbf{I}) are not only unitary, they are also hermitian, meaning that they are identical to their adjoints: $\sigma_j^\dagger = \sigma_j$.

⁴Note that if make the replacements $\theta \rightarrow -\theta$ and $\hat{\mathbf{u}} \rightarrow -\hat{\mathbf{u}}$ we obtain another valid representation of \mathbf{U} , since the physical operation of a rotation by angle θ about an axis $\hat{\mathbf{u}}$ is identical to a rotation by $-\theta$ about the axis $-\hat{\mathbf{u}}$.

There are several unitary operators of interest, such as the Pauli matrices, that are their own inverse. If we call such an operator \mathbf{V} , then its exponential representation of \mathbf{V} can be written in multiple ways,

$$\mathbf{V} = e^{i\delta} e^{i\frac{\delta}{2}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}} = \mathbf{V}^{-1} = e^{-i\delta} e^{-i\frac{\delta}{2}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}}. \quad (14)$$

3. Give the explicit 4×4 matrix form of the four Dirac matrices γ_μ ,⁵ as well as that for $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, in their representation via the 2×2 Pauli matrices \mathbf{I} and $\boldsymbol{\sigma}_i$, $i = 1, 2, 3$,

$$\gamma_0 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0 \end{pmatrix}, \quad (15)$$

It should be then evident that $\text{tr}(\gamma_\mu) = 0 = \text{tr}(\gamma_5)$, where tr is the trace operator. Then, it immediately follows that $\text{tr}(\not{a}) = 0$, where $\not{a} \equiv a^\mu\gamma_\mu$ and a_μ is an arbitrary 4-vector.

Show that

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbf{I}_4, \quad (16)$$

where $\eta_{\mu\nu}$ has diagonal elements $1, -1, -1, -1$ and \mathbf{I}_4 is the 4×4 unit matrix,⁶ and hence that

$$\text{tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}, \quad \text{and} \quad \text{tr}(\not{a}\not{b}) = 4a_\mu b^\mu \equiv 4ab. \quad (17)$$

Show also that

$$\text{tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}), \quad (18)$$

and hence that

$$\text{tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(ab)(cd) - (ac)(bd) + (ad)(bc)]. \quad (19)$$

A factoid which you need not demonstrate is that the Dirac equivalent of eq. (11) is

$$\not{a}\not{b} = ab\mathbf{I}_4 + \frac{a^\mu b^\nu}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \quad (20)$$

If you think that maxtrix manipulation is the key to physics, then you might enjoy my course, **Physics of Quantum Computation**,

<http://physics.princeton.edu/~mcdonald/examples/ph410problems.pdf>.

⁵The matrices γ_μ were introduced by Dirac in the form used here, but with his γ_4 being our γ_0 , in sec. 3 of *The Quantum Theory of the Electron*, Proc. Roy. Soc. London A **117**, 610 (1928), http://physics.princeton.edu/~mcdonald/examples/QED/dirac_prsla_117_610_28.pdf.

⁶The matrix \mathbf{I}_4 is typically denoted by $\mathbf{1}$.

Solutions

1. **Form Factors.** We take the z -axis along the direction of the 3-momentum vector \mathbf{q} , such that the form factor associated with charge density a spherically symmetry charge density $\rho(r)$ is

$$\begin{aligned} F(q^2) &= \int_0^\infty 2\pi r^2 dr \rho(r) \int_{-1}^1 d \cos \theta e^{iqr \cos \theta} = \int_0^\infty 2\pi r^2 dr \rho(r) \frac{e^{iqr} - e^{-iqr}}{iqr} \\ &= \frac{4\pi}{q} \int_0^\infty r dr \rho(r) \sin qr. \end{aligned} \quad (21)$$

Then, the form factor associated with charge density (2) is,

$$\rho(r) = \begin{cases} 3Q/4\pi R^3 & (r < R), \\ 0 & (r > R), \end{cases} \quad (2)$$

$$F_1(q^2) = \frac{3Q}{4\pi R^3} \frac{4\pi}{q^3} \int_0^{qR} qr dqr \sin qr = \frac{3Q}{(qR)^3} (\sin qR - qR \cos qR) \approx Q \left(1 - \frac{(qR)^2}{10} \right). \quad (22)$$

The first zero of F_1 is for $qR = \tan qR$. This can be found by going to Wolfram Alpha, <http://www.wolframalpha.com/>, and entering $x = \tan x$. The result is that $q = 4.493\dots/R$. For $R = 1$ fermi, we need $q = 4.5 \cdot 197 = 877$ MeV to detect the first zero.

The form factor associated with charge density (3) is,

$$\rho(r) = \frac{Q}{4\pi R^2} \delta(r - R), \quad (3)$$

$$F_2(q^2) = \frac{Q}{4\pi R^2} \frac{4\pi}{q} \int_0^\infty r dr \delta(r - R) \sin qr = Q \frac{\sin qR}{qR} \approx Q \left(1 - \frac{(qR)^2}{6} \right). \quad (23)$$

The first zero of F_2 is at $q = \pi/R$.

The form factor associated with charge density (4) is,

$$\rho(r) = \frac{Q}{2\pi\sqrt{2\pi}R^3} e^{-r^2/2R^2}, \quad (4)$$

$$F_3(q^2) = \frac{Q}{2\pi\sqrt{2\pi}R^3} \frac{4\pi}{q} \int_0^\infty r dr e^{-r^2/2R^2} \sin qr = Q e^{-q^2 R^2/2} \approx Q \left(1 - \frac{(qR)^2}{2} \right). \quad (24)$$

F_3 has no zeroes, but its characteristic width in q is $1/R$.

To recover a (spherically symmetric) charge distribution $\rho(r)$ from a form factor $F(q^2)$ we use the inverse transform,

$$\begin{aligned} \rho(r) &= \frac{1}{(2\pi)^3} \int_0^\infty 2\pi q^2 dq F(q^2) \int_{-1}^1 d \cos \theta e^{-iqr \cos \theta} = \frac{1}{(2\pi)^2} \int_0^\infty q^2 dq F(q^2) \frac{e^{-iqr} - e^{iqr}}{-iqr} \\ &= \frac{1}{2\pi^2 r} \int_0^\infty q dq F(q^2) \sin qr. \end{aligned} \quad (25)$$

Hence, the form factor (4) corresponds to

$$\rho_n(r) = \frac{Q}{2\pi^2 r R^{2n}} \int_0^\infty q dq \frac{\sin qr}{[1/R^2 + q^2]^n}. \quad (26)$$

The general form of this integral is given in Gradshteyn and Ryzhik, 3.737.2. For $n = 1$, G&R 3.723.3 gives

$$\rho_1(r) = \frac{Q e^{-r/R}}{4\pi r R^2}, \quad (27)$$

which is not well behaved at the origin, although $\int \rho_1 d\text{Vol} = Q$. For $n = 2$, G&R 3.729.2 gives

$$\rho_2(r) = \frac{Q e^{-r/R}}{8\pi R^3}, \quad (28)$$

a simple exponential falloff. The case $n = 2$ is often called the dipole form factor, for reasons obscure to me. For $n = 3$, G&R 3.737.4 gives

$$\rho_3(r) = \frac{Q e^{-r/R}}{32\pi R^3} \left(1 + \frac{r}{R}\right), \quad (29)$$

which distribution has slightly more charge at large radii than the case of $n = 2$.

2. Arbitrary 2×2 Unitary Matrix

A straightforward expansion of a general 2×2 unitary matrix \mathbf{U} that involves the unit matrix \mathbf{I} and the NOT matrix \mathbf{X} is

$$\begin{aligned} \mathbf{U} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \frac{a+d}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a-d}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{b+c}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{-b+c}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \frac{a+d}{2} \mathbf{I} + \frac{a-d}{2} \mathbf{Z} + \frac{b+c}{2} \mathbf{X} + \frac{-b+c}{2} \tilde{\mathbf{Y}}. \end{aligned} \quad (30)$$

The unitary matrices $\tilde{\mathbf{Y}}$ and \mathbf{Z} have real matrix elements, which seems desirable at first glance. However, when multiplying the unitary matrices based on expansion (30), we find the products

$$\mathbf{X}\tilde{\mathbf{Y}} = \mathbf{Z}, \quad \tilde{\mathbf{Y}}\mathbf{Z} = \mathbf{X}, \quad \mathbf{Z}\mathbf{X} = -\tilde{\mathbf{Y}}. \quad (31)$$

A symmetric pattern of products is obtained, following Pauli, if we use the unitary matrix $\mathbf{Y} = i\tilde{\mathbf{Y}}$. Then,

$$\boldsymbol{\sigma}_x = \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_y = \mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_z = \mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (32)$$

and

$$\mathbf{X}\mathbf{Y} = i\mathbf{Z}, \quad \mathbf{Y}\mathbf{Z} = i\mathbf{X}, \quad \mathbf{Z}\mathbf{X} = i\mathbf{Y}. \quad (33)$$

We can now write our expansion of a general 2×2 unitary matrix as

$$\mathbf{U} = a \mathbf{I} + \mathbf{b} \cdot \boldsymbol{\sigma}, \quad (34)$$

where a is a complex number (in general different from the a of eq. (30)), \mathbf{b} is a triplet of complex numbers, and $\boldsymbol{\sigma}$ is the triplet $(\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z)$ of Pauli matrices.

The (hermitian) Pauli matrices $\boldsymbol{\sigma}_j$ obey

$$\boldsymbol{\sigma}_j^\dagger = \boldsymbol{\sigma}_j, \quad \boldsymbol{\sigma}_j^2 = \mathbf{I}, \quad \text{and} \quad \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k = i \epsilon_{jkl} \boldsymbol{\sigma}_l \quad \text{when } j \neq k, \quad (35)$$

where $\epsilon_{jkl} = 1$ for an even permutation of xyz , -1 for an odd permutation, and 0 otherwise.⁷ Thus,

$$\begin{aligned} (\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) &= \sum_j a_j \boldsymbol{\sigma}_j \sum_k b_k \boldsymbol{\sigma}_k = \sum_{j=k} a_j b_k \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k + \sum_{j \neq k} a_j b_k \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k \\ &= (\mathbf{a} \cdot \mathbf{b}) \mathbf{I} + i \sum_{j \neq k} a_j b_k \epsilon_{jkl} \boldsymbol{\sigma}_l \\ &= (\mathbf{a} \cdot \mathbf{b}) \mathbf{I} + i \boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{b}. \end{aligned} \quad (36)$$

The condition that matrix (34) be unitary can now be written

$$\begin{aligned} \mathbf{I} &= \mathbf{U} \mathbf{U}^\dagger = (a \mathbf{I} + \mathbf{b} \cdot \boldsymbol{\sigma})(a^* \mathbf{I} + \mathbf{b}^* \cdot \boldsymbol{\sigma}^\dagger) = (a \mathbf{I} + \mathbf{b} \cdot \boldsymbol{\sigma})(a^* \mathbf{I} + \mathbf{b}^* \cdot \boldsymbol{\sigma}) \\ &= (|a|^2 + |\mathbf{b}|^2) \mathbf{I} + \boldsymbol{\sigma} \cdot [2\text{Re}(a\mathbf{b}^*) + i \mathbf{b} \times \mathbf{b}^*]. \end{aligned} \quad (37)$$

Hence, we need

$$(|a|^2 + |\mathbf{b}|^2) = 1, \quad (38)$$

$$0 = 2\text{Re}(a\mathbf{b}^*) + i \mathbf{b} \times \mathbf{b}^* = 2\text{Re}(a\mathbf{b}^*) + 2\text{Re}(\mathbf{b}) \times \text{Im}(\mathbf{b}). \quad (39)$$

If $a \neq 0$, we write it as $a = a_0 e^{i\delta}$ where a_0 and δ are real. We also write $\mathbf{b} = e^{i\delta}(\mathbf{c} + i\mathbf{d})$ where \mathbf{c} and \mathbf{d} are real vectors. Then, we eq. (39) becomes

$$0 = \text{Re}(a\mathbf{b}^*) + \text{Re}(\mathbf{b}) \times \text{Im}(\mathbf{b}) = a_0 \mathbf{c} + \mathbf{c} \times \mathbf{d}, \quad (40)$$

which implies that $\mathbf{c} = 0$. Thus,

$$\mathbf{b} = i b_0 e^{i\delta} \hat{\mathbf{u}}, \quad (41)$$

where $b_0 = |d|$ and $\hat{\mathbf{u}} = \mathbf{d}/|d|$ is a real unit vector.

On the other hand, if $a = 0$ then eq. (39) requires that vector $\text{Re}(\mathbf{b})$ must be parallel to vector $\text{Im}(\mathbf{b})$, so the vector \mathbf{b} can be written as

$$\mathbf{b} = \text{Re}(b) \hat{\mathbf{u}} + i \text{Im}(b) \hat{\mathbf{u}} = i b_0 e^{i\delta} \hat{\mathbf{u}}, \quad (42)$$

where b_0 and δ are real, and $\hat{\mathbf{u}}$ is a real unit vector.

⁷**Digression.** If one defines $I = -i\sigma_1$, $J = -i\sigma_2$ and $K = -i\sigma_3$, then $I^2 = J^2 = K^2 = IJK = -\mathbf{I}$, such that $\{\mathbf{I}, I, J, K\}$ are a representation of Hamilton's quaternions (aka Clifford algebra).

Hence, in any case the general 2×2 unitary matrix (34) can be written

$$\mathbf{U} = e^{i\delta}(a_0 \mathbf{I} + ib_0 \hat{\mathbf{u}} \cdot \boldsymbol{\sigma}), \quad (43)$$

where the real numbers a_0 and b_0 obey

$$a_0^2 + b_0^2 = 1, \quad (44)$$

so that condition (38) is satisfied. We can formally express a_0 and b_0 in terms of an angle θ such that

$$a_0 = \cos \frac{\theta}{2}, \quad b_0 = \sin \frac{\theta}{2}. \quad (45)$$

Then,

$$\mathbf{U} = e^{i\delta} \left(\cos \frac{\theta}{2} \mathbf{I} + i \sin \frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma} \right) = e^{i\delta} e^{i\frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma}}. \quad (46)$$

By the exponential $e^{\mathbf{A}}$ of an operator \mathbf{A} we, of course, mean the Taylor series

$$e^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!}. \quad (47)$$

For two noncommuting operators \mathbf{A} and \mathbf{B} , in general $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{B}+\mathbf{A}} \neq e^{\mathbf{A}}e^{\mathbf{B}} \neq e^{\mathbf{B}}e^{\mathbf{A}}$.

The validity of the exponential form in eq. (46) is confirmed by noting that

$$\begin{aligned} e^{i\frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma}} &= \sum_j \frac{(i\frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma})^j}{j!} = \left[\mathbf{I} - \frac{(\frac{\theta}{2})^2 (\hat{\mathbf{u}} \cdot \boldsymbol{\sigma})^2}{2} + \dots \right] + i \left[\frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma} - \frac{(\frac{\theta}{2})^3 (\hat{\mathbf{u}} \cdot \boldsymbol{\sigma})^3}{6} + \dots \right] \\ &= \left[1 - \frac{(\frac{\theta}{2})^2}{2} + \dots \right] \mathbf{I} + i \left[\frac{\theta}{2} - \frac{(\frac{\theta}{2})^3}{6} + \dots \right] \hat{\mathbf{u}} \cdot \boldsymbol{\sigma} = \cos \frac{\theta}{2} \mathbf{I} + i \sin \frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma}, \end{aligned} \quad (48)$$

via repeated uses of eq. (36) with $\mathbf{a} = \mathbf{b} = \hat{\mathbf{u}}$.

While the Pauli operators $\boldsymbol{\sigma}_j$ do not commute with one another, we see from eq. (48) that $e^{a\boldsymbol{\sigma}_j + b\boldsymbol{\sigma}_k} = e^{b\boldsymbol{\sigma}_k + a\boldsymbol{\sigma}_j}$. However, $e^{a\boldsymbol{\sigma}_j + b\boldsymbol{\sigma}_k} \neq e^{a\boldsymbol{\sigma}_j} e^{b\boldsymbol{\sigma}_k} \neq e^{b\boldsymbol{\sigma}_k} e^{a\boldsymbol{\sigma}_j}$ when $j \neq k$. In particular, $e^{i\frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma}} \neq e^{i\frac{\theta}{2} u_x \boldsymbol{\sigma}_x} e^{i\frac{\theta}{2} u_y \boldsymbol{\sigma}_y} e^{i\frac{\theta}{2} u_z \boldsymbol{\sigma}_z}$.

The matrix form of eq. (46) is

$$\mathbf{U} = e^{i\delta} \begin{pmatrix} \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} u_x & i \sin \frac{\theta}{2} (u_x - i u_y) \\ i \sin \frac{\theta}{2} (u_x + i u_y) & \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} u_x \end{pmatrix}, \quad (49)$$

so the determinant of \mathbf{U} is

$$\Delta_{\mathbf{U}} = e^{2i\delta} \left[\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} (u_x^2 + u_y^2 + u_z^2) \right] = e^{2i\delta}. \quad (50)$$

Hence, the 2×2 special unitary operators (those for which $\Delta_{\mathbf{U}} = 1$) are those with $\delta = 0$ or π ,

$$\mathbf{U} = \pm \left(\cos \frac{\theta}{2} \mathbf{I} + i \sin \frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma} \right) = \pm e^{i\frac{\theta}{2} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma}}, \quad \mathbf{U} \in \text{SU}(2). \quad (51)$$

As to the factoid related to eq. (13), whenever $\mathbf{A}^2 = \mathbf{I}$ we can make the Taylor expansion,

$$\begin{aligned} e^{i\theta\mathbf{A}} &= \sum_{k \text{ even}}^{\infty} \frac{(i\theta\mathbf{A})^k}{k!} + \sum_{k \text{ odd}}^{\infty} \frac{(i\theta\mathbf{A})^k}{k!} = \sum_{k \text{ even}}^{\infty} \frac{(-1)^{k/2}\theta^k}{k!} \mathbf{I} + i \sum_{k \text{ odd}}^{\infty} \frac{(-1)^{(k-1)/2}\theta^k}{k!} \mathbf{A} \\ &= \cos \theta \mathbf{I} + i \sin \theta \mathbf{A}. \end{aligned} \quad (52)$$

3. In terms of the 2×2 Pauli matrices \mathbf{I} and $\boldsymbol{\sigma}_i$, the Dirac matrices γ_μ can be written as⁸

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \gamma_1 &= \begin{pmatrix} 0 & \boldsymbol{\sigma}_1 \\ -\boldsymbol{\sigma}_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \gamma_2 &= \begin{pmatrix} 0 & \boldsymbol{\sigma}_2 \\ -\boldsymbol{\sigma}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, & \gamma_3 &= \begin{pmatrix} 0 & \boldsymbol{\sigma}_3 \\ -\boldsymbol{\sigma}_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ \gamma_5 &= i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}. \end{aligned} \quad (53)$$

Note that

$$\begin{aligned} \gamma_0\gamma_0 &= \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{pmatrix} = \mathbf{I}_4, \\ \gamma_i\gamma_i &= \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0 \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i & 0 \\ 0 & -\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i \end{pmatrix} = -\mathbf{I}_4, \\ \gamma_5\gamma_5 &= \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{pmatrix} = \mathbf{I}_4, \end{aligned} \quad (54)$$

⁸The Dirac matrix γ_5 may have been introduced on p. 126 of W. Pauli, *Contributions mathématique à la théorie des matrices de Dirac*, Ann. Inst. H. Poincaré **6**, 109 (1936),

http://physics.princeton.edu/~mcdonald/examples/QED/pauli_ahp_6_109_36.pdf.

Pauli's original γ_5 was just $\gamma_0\gamma_1\gamma_2\gamma_3$, without the factor of i that is now conventional. The matrix $\gamma_0\gamma_1\gamma_2\gamma_3$ was identified as one of 16 linearly independent 4×4 Dirac matrices on p. 881 of J. von Neumann, *Einige Bemerkungen zur Diracschen Theorie des relativistischen Drehelektrons*, Z. Phys. **48** 868 (1928),

http://physics.princeton.edu/~mcdonald/examples/QED/voneumann_zp_48_868_28.pdf.

where \mathbf{I}_4 is the 4×4 unit matrix. Hence, (after a bit of algebra),

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \mathbf{I}_4, \quad \gamma_5 \gamma_\mu = -\gamma_\mu \gamma_5, \quad (55)$$

where we don't regard $\eta_{\mu\nu}$ as a γ -matrix although it is a 4×4 collection of numbers. Taking the trace of this γ -matrix equation, we have that

$$\text{tr}(\gamma_\mu \gamma_\nu) + \text{tr}(\gamma_\nu \gamma_\mu) = 2\text{tr}(\gamma_\mu \gamma_\nu) = 2\eta_{\mu\nu} \text{tr}(\mathbf{I}_4) = 8\eta_{\mu\nu}, \quad (56)$$

$$\text{tr}(\gamma_\mu \gamma_\nu) = 4\eta_{\mu\nu}. \quad (57)$$

Hence,

$$\text{tr}(\not{a}\not{b}) = a^\mu b^\nu \text{tr}(\gamma_\mu \gamma_\nu) = 4a^\mu b^\nu \eta_{\mu\nu} = 4a^\mu b_\nu = 4ab. \quad (58)$$

Next, we consider (following http://en.wikipedia.org/wiki/Gamma_matrices)

$$\text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = \text{tr}(\gamma_\mu \gamma_\nu (2\eta_{\rho\sigma} - \gamma_\sigma \gamma_\rho)) = 2\eta_{\rho\sigma} \text{tr}(\gamma_\mu \gamma_\nu) - \text{tr}(\gamma_\mu \gamma_\nu \gamma_\sigma \gamma_\rho), \quad (59)$$

using eq. (55). For the term on the right, we'll continue the pattern of swapping γ_σ with its neighbor to the left,

$$\text{tr}(\gamma_\mu \gamma_\nu \gamma_\sigma \gamma_\rho) = \text{tr}(\gamma_\mu (2\eta_{\nu\sigma} - \gamma_\sigma \gamma_\nu) \gamma_\rho) = 2\eta_{\nu\sigma} \text{tr}(\gamma_\mu \gamma_\rho) - \text{tr}(\gamma_\mu \gamma_\sigma \gamma_\nu \gamma_\rho). \quad (60)$$

Again, for the term on the right swap γ_σ with its neighbor to the left,

$$\text{tr}(\gamma_\mu \gamma_\sigma \gamma_\nu \gamma_\rho) = \text{tr}((2\eta_{\mu\sigma} - \gamma_\sigma \gamma_\mu) \gamma_\nu \gamma_\rho) = 2\eta_{\mu\sigma} \text{tr}(\gamma_\nu \gamma_\rho) - \text{tr}(\gamma_\sigma \gamma_\mu \gamma_\nu \gamma_\rho). \quad (61)$$

Equation (61) is the term on the right of eq. (60), and eq. (60) is the term on the right of eq. (59). We also use identity (57) to simplify terms, such as,

$$2\eta_{\rho\sigma} \text{tr}(\gamma_\mu \gamma_\nu) = 2\eta_{\rho\sigma} (4\eta_{\mu\nu}) = 8\eta_{\rho\sigma} \eta_{\mu\nu}. \quad (62)$$

So when you plug eqs. (60)-(62) into eq. (59) we have,

$$\text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 8\eta_{\rho\sigma} \eta_{\mu\nu} - 8\eta_{\nu\sigma} \eta_{\mu\rho} + 8\eta_{\mu\sigma} \eta_{\nu\rho} - \text{tr}(\gamma_\sigma \gamma_\mu \gamma_\nu \gamma_\rho). \quad (63)$$

The terms inside the trace can be cycled, so

$$\text{tr}(\gamma_\sigma \gamma_\mu \gamma_\nu \gamma_\rho) = \text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma). \quad (64)$$

So really eq. (62) is

$$2\text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 8\eta_{\rho\sigma} \eta_{\mu\nu} - 8\eta_{\nu\sigma} \eta_{\mu\rho} + 8\eta_{\mu\sigma} \eta_{\nu\rho}, \quad (65)$$

or

$$\text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4(\eta_{\rho\sigma} \eta_{\mu\nu} - \eta_{\nu\sigma} \eta_{\mu\rho} + \eta_{\mu\sigma} \eta_{\nu\rho}). \quad (66)$$

Finally,

$$\begin{aligned} \text{tr}(\not{a}\not{b}\not{c}\not{d}) &= a^\mu b^\nu c^\rho d^\sigma \text{tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4a^\mu b^\nu c^\rho d^\sigma (\eta_{\rho\sigma} \eta_{\mu\nu} - \eta_{\nu\sigma} \eta_{\mu\rho} + \eta_{\mu\sigma} \eta_{\nu\rho}) \\ &= 4[(ab)(cd) - (ac)(bd) + (ad)(bc)]. \end{aligned} \quad (67)$$

Digression: For future reference, we note that

$$\gamma_0 \gamma_i \gamma_5 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}_i & 0 \\ 0 & \boldsymbol{\sigma}_i \end{pmatrix}. \quad (68)$$