Ph 406: Elementary Particle Physics
Problem Set 1

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Due Monday, September 22, 2014 (updated August 19, 2016)

The lecture notes for this course are more or less those at
http://physics.princeton.edu/~mcdonald/examples/index.html#ph529
The problem sets will be posted at http://physics.princeton.edu/~mcdonald/examples/index.html#ph406
Many papers related to this course are in my password-protected directories with links at the top of http://physics.princeton.edu/~mcdonald/examples/
Example: A lecture by Yang at Princeton in 1960 about particle physics just prior to the quark model: http://physics.princeton.edu/~mcdonald/examples/EP/yang_elementary_particles.pdf

1. In this course we will often use "natural" units in which $\hbar = c = 1$, where $\hbar = h/2\pi$ is Planck's constant and c is the speed of light in vacuum. If we also define the unit of mass/energy to be the mass of the proton, what are the units of length and time in this system, expressed in SI units.

   More commonly, we will take the unit of energy to be 1 GeV (or 1 MeV) in our "natural" units. Since the mass of the proton is 0.94 GeV (in these units), the units of length and time in the GeV-$\hbar$-c system differ only by 6% from the values you computed above. That is, a GeV is a rather "natural" unit of energy in a system that emphasizes protons.

2. The cross section σ for the reaction $\pi^- + p \to \Lambda^0 + K^0$ is about 1 mb = 10^{-27} cm^2. Noting that $m_\pi \approx m_p/6$, estimate the dimensionless coupling constant α_S for the (strong) interaction of this process. *Hint: $\sigma \approx \alpha_S^2 \sigma_{\rm geometric}$ in the lowest approximation, which suffices for this problem.*

   The lifetimes τ of the Λ^0 and K_0 particles are both around 10^{-10} s. Noting the $m_\Lambda \approx 2m_K \approx m_p$, estimate the dimensionless coupling constant α_W that is relevant to the (weak) decay processes for these particles.

   What is your estimate for the ratio α_W/α_S of the relative strengths of the weak and strong interactions?

3. Use classical electrodynamics to deduce the Thomson scattering cross section $\sigma_{\gamma e \to \gamma e}$ for the scattering of unpolarized light by an electron nominally at rest. *Hint: Rather than slogging through a derivation based on the differential cross section, as in the text of Jackson, note that $\sigma = P_{\rm scattered}/S_{\rm incident}$, where $\mathbf{S}$ is the Poynting vector and $P_{\rm scattered} = P_{\rm radiated}$ where the latter can be gotten quickly from the so-called Larmor formula. And, it's simpler to use Gaussian units if you are familiar with these.*

4. The so-called quantum electrodynamic critical field strength E_{crit} is that such if an electron were accelerated in this (static, uniform) field for a distance equal to the (reduced) Compton wavelength $\lambda\!\!\!^{-}_C$ of an electron, it would gain energy equal to its rest mass. Deduce an expression for E_{crit} (in Gaussian units), but give a numerical value for it in the hybrid units of volts/cm.

   One also speaks of the critical magnetic field strength, $B_{\text{crit}} = E_{\text{crit}}$. Deduce the value of B_{crit} in gauss, which is the field strength at the magnetic poles of some neutron stars (called magnetars).

   If the QED critical field strength could be achieve, the "vacuum" would "spark," in that "virtual" electron-positron pairs of nominally zero mass would be given enough energy by such a field, while still separated by the size λ_C of the quantum fluctuation for the particles to become "real" with mass/energy mc^2.

   What is the electric field strength at the surface of a lead nucleus, in units of E_{crit}?

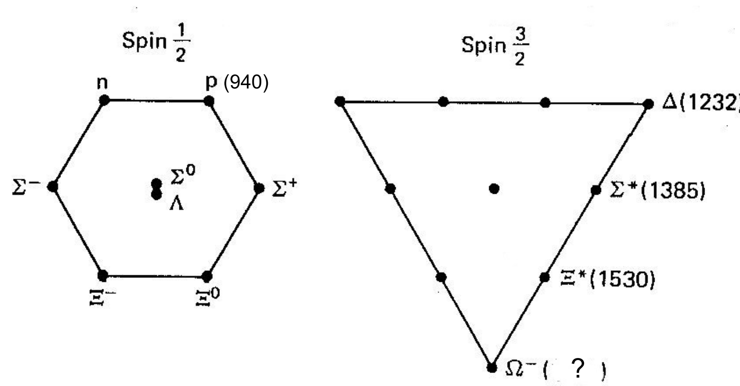
   Note that if a "virtual" electron-positron pair is created (with zero rest energy) out of the vacuum near a nucleus, the electron could be captured into an atomic level with binding energy U, and this energy given to rest energies of the electron and positron. If $U > 2m_ec^2$, then the electron and positron become "real," and we say that the vacuum has "sparked;" otherwise the electron-positron pair must go back into the "vacuum." Use a nonrelativistic Bohr model of an atom with a nucleus of charge Ze to predict the minimum value of Z such that this kind of "sparking the vacuum" could occur. *Relativistic corrections reduce this Z_{crit} significantly. Hint: express the parameter of an atom in terms of $\lambda\!\!\!^{-}_C$, the electromagnetic coupling constant $\alpha_{EM} = \alpha = e^2/\hbar c$, and the electron rest energy m_ec^2.*

   Searches for "sparking the vacuum" in collisions of uranium nuclei, where briefly the total Z is 184, have led to ambiguous results. In an experiment by the author, electron-pairs were produced when a high-energy photon probed a very intense laser beam, whose electric field strength was close E_{crit} in the rest frame electron-positron pair; the results can be interpreted in the complementary ways of "sparking the vacuum" or the nonlinear reaction $\gamma + n\gamma_{\text{laser}} \to e^+e^-$. See sec. IVb of C. Bamber et al., Phys. Rev. D **60***, 092004 (1999),*

   http://physics.princeton.edu/~mcdonald/examples/QED/bamber_prd_60_092004_99.pdf

   Note that a strong laser beam (plane electromagnetic wave) cannot by itself "spark the vacuum" in that an electron-positron pair has a rest frame, while there is no rest frame for a collection of identical photons.

5. Using the information given in the diagram of the baryon octet and decuplet, with masses in MeV, predict the (constituent) masses of the u, d and s quarks, and the mass of the Ω^- baryon.
The latter prediction was the only one from the quark model that was verified between its development and its Nobel Prize.



6. What would the characteristic binding energy and radius of a possible neutron-electron atom (bound state) as a result of the force between their magnetic moments?

You could recall certain general theorems of classical mechanics, and/or give a semi-classical argument in the spirit of Bohr, supposing that the two magnetic moments are (anti)parallel. The magnitudes of the magnetic moments can be written as

$$\mu_e = g_e \frac{e\hbar}{2m_e c} = \frac{g_2}{2} e \lambda\!\!\!^-_e \,, \qquad \mu_n = g_n \frac{e\hbar}{2m_n c} = \frac{g_n}{2} \frac{m_e}{m_n} e \lambda\!\!\!^-_e \,, \tag{1}$$

where $g_e \approx 2$, $g_n \approx 2.8$, e is the magnitude of the charge of the electron, m_e and m_n are the masses of the electron and neutron, c is the speed of light, and $\lambda\!\!\!^-_e = \hbar/m_e c \approx 3.9 \times 10^{-11}$ cm.

Solutions

1. The dimension of c are $[c] = [l]/[t]$, and the dimensions of $\hbar$ are those of action = energy × time, $[\hbar] = [m][l]^2/[t]$. Thus,

$$[l] = \frac{[\hbar]}{[c][m]}\,, \qquad \text{and} \qquad [t] = \frac{[l]}{[c]}\,. \tag{2}$$

In SI units, $c \approx 3 \times 10^8$ m/s, $\hbar \approx 1.05 \times 10^{-34}$ m^2 kg/s, and the mass of the proton is $m_p = 1.67 \times 10^{-27}$ kg.

If the also define the unit of mass to be m_p, then the unit of length in this system is

$$\frac{\hbar}{c\,m_p} \approx \frac{1.05 \times 10^{-34}\ \mathrm{m^2\,kg/s}}{3 \times 10^8\ \mathrm{m/s} \cdot 1.67 \times 10^{-27}\ \mathrm{kg}} \approx 2.1 \times 10^{-16}\ \mathrm{m} = 0.21\ \mathrm{fermi}, \tag{3}$$

and the unit of time is

$$\frac{2.1 \times 10^{-16}\ \mathrm{m}}{3 \times 10^8\ \mathrm{m/s}} = 7 \times 10^{-25}\ \mathrm{s}. \tag{4}$$

2. The geometric cross section for scattering of π^- and p is $\sigma_{\text{geometric}} = \pi(r_\pi + r_p)^2$. From prob. 1, we have that the radius of a proton is $r_p \approx 0.2$ fermi $= 0.2 \times 10^{-13}$ cm. Since length is inversely proportional to mass, we estimate that

$$r_\pi \approx r_p \frac{r_p}{m_\pi} \approx 6 r_p \approx 1.2\ \mathrm{fermi}. \tag{5}$$

Hence,

$$\sigma_{\text{geometric}} \approx \pi(1.2 + 0.2)^2\ \mathrm{fermi}^2 \approx 6 \times 10^{-26}\ \mathrm{cm}^2 \approx 60\,\sigma, \tag{6}$$

for $\sigma \approx 10^{-27}$ cm^2. We then estimate the coupling constant as

$$\alpha_S \approx \sqrt{\frac{\sigma}{\sigma_{\text{geometric}}}} \approx \frac{1}{8}\,. \tag{7}$$

In prob. 1 we saw that the characteristic time associated with m_p is $\tau_0 \approx 10^{-23}$ s, which would be the lifetime of particles of this mass if the dimensionless decay constant were 1. So, we estimate the relevant coupling constant for Λ^0 and K^0, whose masses are roughly m_p, to be

$$\tau \approx \tau_0\,\alpha_W^2, \qquad \alpha_W \approx \sqrt{\frac{\tau_0}{\tau}} \approx \sqrt{\frac{10^{-23}}{10^{-10}}} \approx 3 \times 10^{-7}. \tag{8}$$

Then, $\alpha_W/\alpha_S \approx 2.5 \times 10^{-6}$, which is a reasonable estimate of the relative strengths of the weak and strong interactions.

3. It is easier to use Gaussian units to discuss classical electrodynamics of waves. Then, the Poynting vector is given by

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}, \qquad \text{such that} \qquad S_{\rm in} = \frac{cE_{\rm in}B_{\rm in}}{4\pi} = \frac{cE_{\rm in}^2}{4\pi}\,. \tag{9}$$

The scattered power is given by the (memorable) Larmor formula,

$$P_{\rm rad} = \frac{2e^2a^2}{3c^3}\,, \tag{10}$$

where the acceleration of an electron by the incident wave is $a = eE_{\rm in}/m_e$. Thus, $P_{\rm scat} = P_{\rm rad} = 2e^4E_{\rm in}^2/3m_e^2c^3$, and the scattering cross section is

$$\sigma_{\rm Thomson} = \frac{P_{\rm scat}}{S_{\rm in}} = \frac{8\pi/}{3}\left(\frac{e}{m_ec^2}\right)^2 = \frac{8\pi r_e^2}{3} \approx 7\times 10^{-25}\ {\rm cm}^2, \tag{11}$$

where $r_e = e^2/m_ec^2 \approx 3$ fermi is the classical electron radius.

4. Recalling that the reduced Compton wavelength of an electron is $\lambda\!\!\!^{-}_e = \hbar/m_ec$, the work done on an electron that moves distance $\lambda\!\!\!^{-}_e$ due to an external electric field E is $W = eE\lambda\!\!\!^{-}_e$. Setting this equal to the rest mass/energy of an electron, we have that $eE_{\rm crit}\hbar/mc = mc^2$, and hence,

$$E_{\rm crit} = \frac{m^2c^3}{e\hbar}\,. \tag{12}$$

Since

$$\lambda\!\!\!^{-}_e = \frac{\hbar}{m_ec} = \frac{e^2/m_ec^2}{e^2/\hbar c} = \frac{r_e}{\alpha} \approx \frac{3\times 10^{-13}\ {\rm cm}}{1/137} \approx 4\times 10^{-11}\ {\rm cm}, \tag{13}$$

where $\alpha = e^2/\hbar c = 1/137$ is the so-called fine-structure constant, and $m_ec^2 = 0.511$ MeV, we have that

$$E_{\rm crit} = \frac{m_ec^2}{e\lambda\!\!\!^{-}} = \frac{0.511\ {\rm MeV}}{4\times 10^{-11}e - {\rm cm}} \approx 1.3\times 10^{10}\ {\rm MV/cm} = 1.3\times 10^{16}\ {\rm V/cm}. \tag{14}$$

The QED critical magnetic field $B_{\rm crit}$ has the same value as $E_{\rm crit}$ in Gaussian units, but volts is not a Gaussian unit. Recall that the Gaussian unit of voltage is the statvolt, where 1 statvolt = 300 volts. Hence,

$$B_{\rm crit} \approx 1.3\times 10^{16}\ {\rm V/cm} = \frac{1.3\times 10^{16}}{300}\ {\rm statvolt/cm} \approx 4.4\times 10^{13}\ {\rm gauss}. \tag{15}$$

The electric field strength at the surface of a lead nucleus is

$$\begin{aligned} E_{\rm lead} &= \frac{Z_{\rm lead}e}{r_{\rm lead}^2} = \frac{Z_{\rm lead}e}{A_{\rm lead}^{2/3}r_p^2} \approx \frac{82e}{207^{2/3}\cdot(1\ {\rm fermi})^2} \approx \frac{82\cdot 4.8\times 10^{-10}}{35\cdot 10^{-26}} \\ &\approx 1.12\times 10^{17}\ {\rm statvolt/cm} \approx 3.37\times 10^{19}\ {\rm volt/cm} \approx 2000E_{\rm crit}, \end{aligned} \tag{16}$$

noting that the charge of an electron/proton is 4.8×10^{-10} statcoulomb.

Or, computing is SI units,

$$\begin{aligned} E_{\text{lead}} &= \frac{Z_{\text{lead}} e}{4\pi\epsilon_0 r_{\text{lead}}^2} = \frac{Z_{\text{lead}} e}{4\pi\epsilon_0 A_{\text{lead}}^{2/3} r_p^2} = \frac{82e}{4\pi\epsilon_0 \cdot 207^{2/3} \cdot (1 \text{ fermi})^2} \\ &\approx \frac{9 \times 10^9 \cdot 82 \cdot 1.6 \times 10^{-19}}{35 \cdot 10^{-30}} \approx 3.37 \times 10^{21} \text{ volt/m} = 3.37 \times 10^{19} \text{ volt/cm}, \end{aligned} \tag{17}$$

noting that the charge of an electron/proton is 1.6×10^{-19} coulomb and that $1/4\pi\epsilon_0 \approx 9 \times 10^9$ SI units.

In the semiclassical model of Bohr, we consider a (point) nucleus of charge Ze, such that for an electron in a circular orbit of radius r Newton's 2nd law gives

$$\frac{m_e v^2}{r} = \frac{Ze^2}{r^2}. \tag{18}$$

Bohr's quantum condition was that for the lowest energy state, the angular momentum would be

$$L = mvr = \hbar. \tag{19}$$

This constrains the radius r_0 of the lowest energy state to be

$$r_0 = \frac{\hbar^2}{Ze^2 m_e} = \frac{1}{Z}\frac{\hbar c}{e^2}\frac{\hbar}{m_e c} = \frac{\lambda\!\!\!^{-}_c}{Z\alpha} = \frac{r_e}{Z\alpha^2}, \tag{20}$$

such that the energy is

$$U = \frac{m_e v^2}{2} - \frac{Ze^2}{r_0} = -\frac{Ze^2}{2r_0} = -\frac{Z^2 e^2 \alpha}{2\lambda\!\!\!^{-}_C} = -\frac{Z^2 e^2 \alpha m_e c}{2\hbar} = -\frac{Z^2 \alpha^2 m_e c^2}{2}. \tag{21}$$

To have $|U| = 2m_e c^2$ requires that

$$Z_{\text{crit}} = \frac{2}{\alpha} = 274, \tag{22}$$

in which case $r_0 < \lambda\!\!\!^{-}_C$ and $E(r_0) = Ze/r_0^2 = Z^3 e^5 m_e^2/\hbar^4 = (Z\alpha)^3 m_e^2 c^3/e\hbar = 8E_{\text{crit}}$.

The relativistic variant of the Bohr atom given by Sommerfeld predicted the energy of a K-electron to be

$$U = -\left(1 - \frac{1}{\sqrt{1 + (Z\alpha)^2 / \left[n_r + \sqrt{n_\phi^2 - (Z\alpha)^2}\right]^2}}\right) mc^2, \tag{23}$$

with $n_r = 0$ and $n_\phi = 1$, which expression approaches mc^2 as Z approaches $1/\alpha$ and is complex for larger Z. An identical expression is obtained for a Dirac electron in the Coulomb potential of a point nucleus of charge Ze, although the interpretation of the quantum numbers n_r and n_ϕ is different ($n_\phi = j + 1/2$, $n_r = n - j - 1/2$). These models predict that $Z_{\text{crit}} = 1/\alpha \approx 137$ (for a point nucleus).

5. Since the proton is a *uud* and the neutron is a *udd* 3-quark state, we infer that the (constituent) masses of the u- and d-quarks are each about 310 MeV. The Σ, Ξ and the Ω baryons has 1, 2 and 3 s-quarks in place of u- or d-quarks. From the masses in the baryon decuplet, we infer that the s-quark weights about 150 MeV more than the u- and d-quarks, and we predict that the Ω^- baryon will have a mass of 1680 MeV.

   Its mass is 1672 MeV.
   http://physics.princeton.edu/~mcdonald/examples/EP/barnes_prl_12_204_64.pdf

6. The classical potential energy U of a pair of magnetic moments $\boldsymbol{\mu}_n$ and $\boldsymbol{\mu}_e$ of fixed magnitude is

$$U = -\boldsymbol{\mu}_e \cdot \mathbf{B}_n = \frac{\boldsymbol{\mu}_e \cdot \boldsymbol{\mu}_n - 3(\boldsymbol{\mu}_n \cdot \hat{\mathbf{r}})(\boldsymbol{\mu}_n \cdot \hat{\mathbf{r}})}{r^3} \to -\frac{\mu_e \mu_n (1 - 3\cos^2\theta)}{r^3}, \tag{24}$$

   for antiparallel moments, where angle θ is measured with respect to the direction of the moments. For motion in the plane perpendicular to that direction, this potential corresponds to an attractive force, so a bound state of an electron and neutron might exist.

   However, such a state could not be spherically symmetric (S-wave), since in that case the average potential energy is zero,

$$\langle U \rangle \propto -\frac{\mu_e \mu_n}{r^3} \int_{-1}^{1} (1 - 3\cos^2\theta)\, d\cos\theta = 0. \tag{25}$$

   Hence, possible bound states must carry orbital angular momentum, the smallest nonzero value of which is

$$\hbar = L = pr = \frac{m_e c r (v/c)}{\sqrt{1 - (v/c)^2}}, \qquad \frac{c}{v} = \sqrt{1 + (r/\lambda\!\!\!^{-}_e)^2}, \tag{26}$$

   where we approximate the neutron as being at rest with the electron in an orbit of radius r with momentum p, and with the magnetic moments perpendicular to the plane of the orbit, and $\lambda\!\!\!^{-}_e = \hbar/m_e c = 3.9 \times 10^{-11}$ cm is the (reduced) Compton wavelength of the electron.

   Them, the magnetic force on the electron is

$$\mathbf{F} = -\boldsymbol{\nabla} U = \boldsymbol{\nabla} \frac{\mu_e \mu_n}{r^3} = -\frac{3\mu_e \mu_n}{r^4}\,\hat{\mathbf{r}}. \tag{27}$$

   At this point we could recall the so-called virial theorem of mechanics,[1] which tells us that bound states in central potentials $U = -K/r^n$ exist (at least for nonrelativistic motion) only for $n < 2$. In particular, for $n = 3$, the virial theorem states that $\langle T \rangle = -3\langle U \rangle/2$, $\langle T + U \rangle = -\langle U \rangle/2 > 0$, where T is the kinetic energy. Hence, it seems likely that no electron-neutron bound state exists. However, it might be that the motion is relativistic, so we continue the discussion.

[1] R. Clausius, *On a mechanical theorem applicable to heat*, Phil. Mag. **40**, 122 (1879), http://physics.princeton.edu/~mcdonald/examples/mechanics/clausius_pm_40_122_70.pdf.

The force (27) equals the rate of change of the electron's momentum $\mathbf{p}$, which is possibly relativistic,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \boldsymbol{\omega} \times \mathbf{p} = -p\omega\,\hat{\mathbf{r}} = -\frac{pv}{r}\,\hat{\mathbf{r}}, \tag{28}$$

where v is the velocity of the electron. Equating (27) and (28) leads to Kepler's law of the electron-neutron system,

$$r^3 = \frac{3\mu_e\mu_n}{pv} = \frac{3g_e g_n e^2\hbar^2}{4m_e m_n c^2 pv} = \frac{3g_e g_n}{4}\frac{e^2}{\hbar c}\left(\frac{\hbar}{m_e c}\right)^2 \frac{m_e}{m_n}\frac{\hbar c}{pv} = \frac{3\alpha g_e g_n \lambda\!\!\!^-_e{}^2}{4}\frac{m_e}{m_n}\frac{\hbar c}{pv}\,, \tag{29}$$

where $\alpha = e^2/\hbar c = 1/137$.

Using the quantum condition (26), we have that

$$\frac{r^2}{\lambda\!\!\!^-_e{}^2} = \frac{3\alpha g_e g_n}{4}\frac{m_e}{m_n}\sqrt{1 + (r/\lambda\!\!\!^-_e)^2} \approx \frac{3\alpha g_e g_n}{4}\frac{m_e}{m_n}\,, \tag{30}$$

$$\frac{r^2}{\lambda\!\!\!^-_e{}^2} \approx 1.3 \times 10^{-5}, \qquad r \approx 0.037\lambda\!\!\!^-_e = 1.4 \times 10^{-14}\ \text{cm}. \tag{31}$$

This is smaller than the charge radius of the neutron, $\approx 1 \times 10^{-13}$ cm, so it is implausible that a stable bound state of an electron and neutron exists.

We now see that the motion of the electron is indeed relativistic, with $v \approx c$, such that the total mechanical energy of the electron is

$$E_e \approx pc \approx \frac{3\mu_e\mu_n}{r^3}\,, \tag{32}$$

recalling the first form of eq. (30). The magnetic potential energy at this radius is

$$U = -\frac{\mu_e\mu_n}{r^3} \approx -\frac{E}{3}\,, \tag{33}$$

such that the total energy of the system is

$$\begin{aligned} E_{\text{total}} &= m_n c^2 + E_e + U \approx m_n c^2 + \frac{2\mu_e\mu_n}{r^3} \approx m_n c^2\left(1 + \frac{1}{3}\sqrt{\frac{2m_e}{3\alpha g_e g_n m_n}}\right) \approx 1.03 m_n c^2 \\ &= 967\ \text{MeV} > (m_n + m_e)c^2. \end{aligned} \tag{34}$$

Such a state is not bound, as anticipated by Clausius' nonrelativistic virial theorem, but might appear as a 28-MeV resonance in e-n scattering.

Historically, a report of a possible electron-neutron bound state in 1969 was quickly contradicted by other experiments. For commentary on this, see, for example, Schlitt[2] and references therein.

For the related example of an electric charge together with an electric dipole, see http://physics.princeton.edu/~mcdonald/examples/dipole.pdf.

[2]D.W. Schlitt, *Comment on "Does a Free Neutron-Electron Bound State Exist?"* Am. J. Phys. **41**, 1120 (1973), http://physics.princeton.edu/~mcdonald/examples/QM/schlitt_ajp_41_1120_73.pdf.