

Volume and Surface Area of an N -Sphere

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1 Problem

Deduce expressions for the volume and surface area of a (Euclidean) N -sphere.

2 Solution

This solution follows <http://db.uwaterloo.ca/~alopez-o/math-faq/node75.html> by A. Lopez-Ortiz. Who first gave this solution?

We expect that the volume V_N of an N -sphere varies with its radius r as

$$V_N = C_N r^N, \quad (1)$$

where the C_N are constants to be determined.

If we consider the N -sphere to be made up of a set of concentric shells, then the volume dV_N of a shell of radius r and thickness dr is related the surface area A_N of the shell by

$$dV_N = A_N dr. \quad (2)$$

Thus,

$$A_N = \frac{dV_N}{dr} = N C_N r^{N-1}. \quad (3)$$

A clever method to evaluate the C_N is to consider the integral of e^{-r^2} in both rectangular and “spherical” coordinates:

$$\begin{aligned} \int e^{-r^2} dV_N &= \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_N e^{-x_1^2 - \dots - x_N^2} = \left(\int_{-\infty}^{\infty} dx e^{-x^2} \right)^N = \pi^{N/2} \\ &= \int_0^{\infty} e^{-r^2} A_N dr = N C_N \int_0^{\infty} e^{-r^2} r^{N-1} dr = \frac{N C_N}{2} \int_0^{\infty} e^{-s} s^{N/2-1} ds \\ &= \frac{N C_N}{2} \Gamma(N/2) = \frac{N C_N}{2} (N/2 - 1)! = \frac{N C_N}{2} \frac{(N/2)!}{N/2} = C_N (N/2)! \end{aligned} \quad (4)$$

Thus,

$$C_N = \frac{\pi^{N/2}}{(N/2)!}, \quad (5)$$

so the volume and surface area of an N -sphere are

$$V_N = \frac{\pi^{N/2}}{(N/2)!} r^N, \quad A_N = N \frac{\pi^{N/2}}{(N/2)!} r^{N-1}. \quad (6)$$

An expression for $(N/2)!$ for odd integer N can be deduced from the fact that

$$\Gamma(1/2) = \int_0^\infty e^{-s} s^{-1/2} ds = 2 \int_0^\infty e^{-r^2} dr = \sqrt{\pi} \quad (7)$$

and the recurrence relation

$$\Gamma(x+1) = x\Gamma(x). \quad (8)$$

Thus,

$$\begin{aligned} (N/2)! &= \Gamma(N/2 + 1) = \sqrt{\pi} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{N}{2} = \sqrt{\pi} \cdot \frac{1}{2} \cdot \frac{2}{2 \cdot 1} \cdot \frac{3}{2} \cdot \frac{4}{2 \cdot 2} \cdots \frac{N}{2} \cdot \frac{N+1}{2 \cdot \frac{N+1}{2}} \\ &= \sqrt{\pi} \frac{(N+1)!}{\left(\frac{N+1}{2}\right)! 2^{N+1}} \quad (\text{odd } N). \end{aligned} \quad (9)$$

With this, we find the first few C_N to be

$$C_1 = 2, \quad C_2 = \pi, \quad C_3 = \frac{4}{3}\pi, \quad C_4 = \frac{\pi^2}{2}, \quad C_5 = \frac{8\pi^2}{15}, \quad C_6 = \frac{\pi^3}{6}, \quad C_7 = \frac{16\pi^3}{105}, \quad C_8 = \frac{\pi^4}{24}. \quad (10)$$

For large N , Stirling's approximation yields

$$C_N \approx \frac{1}{\sqrt{N\pi}} \left(\frac{2e\pi}{N}\right)^{N/2} \quad (N \gg 1). \quad (11)$$