

Radiated-Power Distribution in the Far Zone of a Moving System

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(April 24, 1979; updated April 28, 2017)

1 Problem

Show that the angular distribution,

$$\frac{dP^*}{d\Omega^*} = \frac{d^2U^*}{dt^* d\Omega^*} = f(\cos \theta^*, \phi^*), \quad (1)$$

of the power of electromagnetic radiation in the far zone of a system whose center of mass/energy is instantaneously at rest in the (inertial) $*$ frame has the form,

$$\frac{dP_{\text{source}}}{d\Omega} = \frac{dU}{dt d\Omega} = \frac{1}{\gamma^4(1 - \beta \cos \theta)^3} f\left(\frac{\cos \theta - \beta}{1 - \beta \cos \theta}, \phi\right), \quad (2)$$

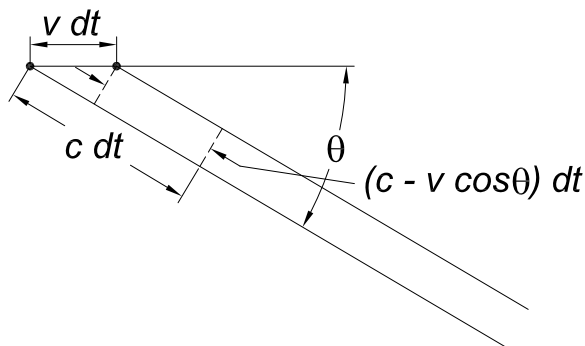
for radiation by the source in the (inertial) lab frame where the system has velocity \mathbf{v} along the polar (z, z^*) axes of the spherical coordinate systems (r, θ, ϕ) and (r^*, θ^*, ϕ^*) , $\beta = v/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$, with c being the speed of light in vacuum. Comment on the angular distribution of radiation as detected by distant, fixed observers in the lab frame.

Comment also on the case of acoustic radiation.

2 Solution

2.1 Effect of Retardation in the Lab Frame

While the main theme of this problem is the transformation between frames of power emitted by a source, we first take note of the distinct, but related issue that the power emitted by a moving source into some solid angle, as measured in the lab frame, is different from that received by a fixed observer (in the lab frame) who subtends the same solid angle.



As suggested by the sketch above, the energy dU_{source} emitted by a moving source at angle θ with respect to its velocity \mathbf{v} during time interval dt is compressed into the spatial

interval $(c - v \cos \theta)dt$ along the direction of emission, where c is the speed of propagation in the lab frame. This energy is later received by a fixed observer during the time interval $(1 - v \cos \theta/c)dt$, such that the power received is related by,

$$\frac{dP_{\text{observer}}}{d\Omega} = \frac{1}{(1 - v \cos \theta/c)} \frac{d^2 U_{\text{source}}}{dt d\Omega} = \frac{1}{(1 - v \cos \theta/c)} \frac{dP_{\text{source}}}{d\Omega}. \quad (3)$$

The relation (3) can also be considered as a consequence of the fact that the energy received by the observer (in solid angle $d\Omega$) during time interval dt was emitted by the source during the retarded time interval dt' (into the same solid angle $d\Omega$), where,¹

$$t' = t - \frac{r(t')}{c}, \quad t = t' + \frac{r(t')}{c}, \quad dt = dt' \left(1 + \frac{1}{c} \frac{dr}{dt'}\right) = dt' \left(1 - \frac{v}{c} \cos \theta\right), \quad (4)$$

and hence,

$$\frac{dP_{\text{observer}}}{d\Omega} = \frac{d^2 U}{dt d\Omega} = \frac{1}{(1 - v \cos \theta/c)} \frac{d^2 U}{dt' d\Omega} = \frac{1}{(1 - v \cos \theta/c)} \frac{dP_{\text{source}}}{d\Omega}, \quad (5)$$

as found previously.

Both t and t' are measured in the lab frame, so the difference between P_{source} and P_{observer} (both in the lab frame) is not an effect of “relativity”, but can be attributed to retardation.

In the instantaneous rest frame of the source, where $\beta^ = 0$, the power radiated by the source in some solid angle is the same as that detected by a fixed observer with that solid angle.*

2.2 Electromagnetic Radiation

2.2.1 General Transformation of the Angular Distribution

The arguments here are based on the semiclassical insight that far from its source, the energy and momentum densities u and \mathbf{p} of electromagnetic radiation (in vacuum) are related by,

$$u = pc, \quad (6)$$

as for a single photon. Hence, we can use a light-like 4-vector $(du, c d\mathbf{p})$ to describe the propagating energy and momentum of a small volume element (which moves at the speed of light) in the far zone.

We take the z -axis to be along the direction of the velocity \mathbf{v} of the source at some (retarded) time in the lab frame, and let θ be the angle between the momentum vector $d\mathbf{p}$ and the z -axis. Then, the energy and momentum densities in the instantaneous rest frame (the \star frame) of the (retarded) source are given by,

$$du^{\star} = \gamma(du - \mathbf{v} \cdot d\mathbf{p}) = \gamma du(1 - \beta \cos \theta), \quad (7)$$

$$dp_{\perp}^{\star} = dp_{\perp} = \frac{du}{c} \sin \theta, \quad (8)$$

$$dp_{\parallel}^{\star} = \gamma(dp_{\parallel} - v du/c^2) = \gamma \frac{du}{c} (\cos \theta - \beta). \quad (9)$$

¹The quantity $1 - v \cos \theta/c$ is sometimes called the Doppler factor. Its early history, with some initial controversy over its interpretation can be traced in [1, 2, 3, 4, 5]. This factor was well discussed by Schott, sec. 13 of [6], but apparently needed “rediscovering” in 1969, as on pp. 406-407 of [7]. *The author learned of this factor in graduate school (Caltech) in 1968, where it was presented as “well known”.*

The angle θ^* of the vector $d\mathbf{p}^*$ with respect to the z/z^* axis is given by,

$$\cos \theta^* = \frac{dp_{\parallel}^*}{p^*} = \frac{dp_{\parallel}^*}{\sqrt{p_{\parallel}^{*2} + p_{\perp}^{*2}}} = \frac{\cos \theta - \beta}{\sqrt{(\cos \theta - \beta)^2 + (1 - \cos^2 \theta)/\gamma^2}} = \frac{\cos \theta - \beta}{1 - \beta \cos \theta}, \quad (10)$$

and hence,

$$d \cos \theta^* = \frac{d \cos \theta}{\gamma^2 (1 - \beta \cos \theta)^2}. \quad (11)$$

Of course, the azimuthal angles ϕ and ϕ^* have the same values, so the transformation of solid angle $d\Omega = d \cos \theta d\phi$ is,

$$d\Omega^* = \frac{d\Omega}{\gamma^2 (1 - \beta \cos \theta)^2}. \quad (12)$$

Also, a time interval dt for a fixed observer in the lab frame is related to the time interval dt^* in the moving $*$ frame by time dilation,

$$dt = \gamma dt^*. \quad (13)$$

Combining eqs. (7) and (10)-(13), the angular distribution of the power radiated by the moving source can be written as,

$$\frac{dP_{\text{source}}}{d\Omega} = \frac{d^2 u}{dt d\Omega} = \frac{1}{\gamma^4 (1 - \beta \cos \theta)^3} \frac{d^2 u^*}{dt^* d\Omega^*} = \frac{1}{\gamma^4 (1 - \beta \cos \theta)^3} f\left(\frac{\cos \theta - \beta}{1 - \beta \cos \theta}, \phi\right). \quad (14)$$

This result is based on the transformation of the energy and momentum far from the source, and so corresponds to the angular distribution of power according to a distant observer. According to the argument of sec. 2.1, the angular distribution of power as detected by a fixed, distant observer is,

$$\frac{dP_{\text{observer}}}{d\Omega} = \frac{1}{1 - \beta \cos \theta} \frac{dP_{\text{source}}}{d\Omega} = \frac{1}{\gamma^4 (1 - \beta \cos \theta)^4} f\left(\frac{\cos \theta - \beta}{1 - \beta \cos \theta}, \phi\right). \quad (15)$$

For example, although a single electric charge cannot emit radiation with an isotropic angular distribution, this is possible for an appropriate current distribution [8]. If that current distribution had bulk velocity \mathbf{v} , the angular distribution observed in the lab frame would be,

$$\frac{dP_{\text{observer}}}{d\Omega} = \frac{1}{\gamma^4 (1 - \beta \cos \theta)^4}, \quad (16)$$

and the forward/backward ratio would be,

$$\frac{dP_{\text{observer}}(\theta = 0)/d\Omega}{dP_{\text{observer}}(\theta = \pi)/d\Omega} = \frac{(1 + \beta)^4}{(1 - \beta)^4} \approx \begin{cases} 1 + 8\beta & (\beta \ll 1), \\ 32\gamma^8 & (\gamma \gg 1, \beta \approx 1 - 1/2\gamma^2). \end{cases} \quad (17)$$

2.2.2 Radiation by a Single Accelerated Charge

If an electric charge e has acceleration \mathbf{a}^* in its instantaneous rest frame, then its angular distribution of radiated power in that frame is given by the differential version of the Larmor formula (in Gaussian units),

$$\frac{dP^*}{d\Omega^*} = \frac{e^2 a^{*2} \sin^2 \alpha^*}{4\pi c^3}, \quad (18)$$

where α^* is the angle between the acceleration vector and the direction to the observer in the * frame.

To relate the lab-frame acceleration 3-vector \mathbf{a} to the acceleration \mathbf{a}^* in its instantaneous rest frame we recall that a velocity 4-vector can be defined as,

$$v_\mu = \gamma(c, \mathbf{v}) = c\gamma(1, \boldsymbol{\beta}), \quad (19)$$

and then the acceleration 4-vector is given by,

$$a_\mu = \frac{dv_\mu}{d\tau} = \gamma \frac{dv_\mu}{dt} = c\gamma^2(\gamma^2(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}), \dot{\boldsymbol{\beta}} + \gamma^2\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}})), \quad (20)$$

noting that,

$$\frac{d\gamma}{dt} = \gamma^3(\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}), \quad (21)$$

where $\dot{\boldsymbol{\beta}} = d\boldsymbol{\beta}/dt$. The acceleration 4-vector in the instantaneous rest frame has components $c(0, \dot{\boldsymbol{\beta}}^*) = (0, \mathbf{a}^*)$, so the invariant square of the acceleration 4-vector is,

$$a_\mu a^\mu = -a^{*2} = -c^2\gamma^6[\dot{\boldsymbol{\beta}}^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2] = -\gamma^6[a^2 - (\mathbf{v}/c \times \mathbf{a})^2]. \quad (22)$$

Using eqs. (18) and (22) in (14), we the power radiated by the accelerated charge is,

$$\frac{dP_{\text{source}}}{d\Omega} = \frac{\gamma^2 e^2 [a^2 - (\mathbf{v}/c \times \mathbf{a})^2]}{4\pi c^3 (1 - \beta \cos \theta)^3} \sin^2 \alpha^*(\theta, \phi). \quad (23)$$

The power radiated by an accelerated charge, as detected by a distant, fixed observer, is often computed from the Poynting vector of the Lienard-Wiechert fields of an accelerated charge, yielding,²

$$\frac{dP_{\text{observer}}}{d\Omega} = r^2 S_{\text{far}} = \frac{r^2 E_{\text{far}}^2}{4\pi c} = \frac{e^2 [\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \mathbf{a})]^2}{4\pi c^3 (1 - \beta \cos \theta)^6}, \quad (24)$$

$$\frac{dP_{\text{source}}}{d\Omega} = (1 - \beta \cos \theta) \frac{dP_{\text{observer}}}{d\Omega} = \frac{e^2 [\hat{\mathbf{n}} \times ((\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \mathbf{a})]^2}{4\pi c^3 (1 - \beta \cos \theta)^5}, \quad (25)$$

where $\hat{\mathbf{n}}$ is a unit vector along the direction from the (retarded) source to the observer in the lab frame. The forms (23) and (25) are equivalent, despite their apparent differences.

²See, for example, sec. 14.3 of [9].

2.2.3 Acceleration Parallel to the Velocity

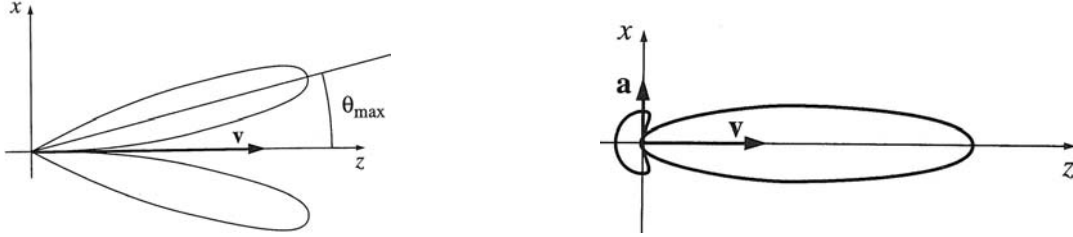
When the acceleration \mathbf{a} is parallel to the velocity \mathbf{v} , then $a^{*2} = \gamma^6 a^2$ and the polar angle θ^* is the same as angle α^* . Then,

$$\sin^2 \alpha^* = \sin^2 \theta^* = 1 - \cos^2 \theta^* = 1 - \frac{(\cos \theta - \beta)^2}{(1 - \beta \cos \theta)^2} = \frac{(1 - \beta^2) \sin^2 \theta}{(1 - \beta \cos \theta)^2} = \frac{\sin^2 \theta}{\gamma^2 (1 - \beta \cos \theta)^2}, \quad (26)$$

and radiated power in the lab frame follows from eqs. (14) and (18), or from eqs. (23)-(24), as,

$$\frac{dP_{\parallel, \text{source}}}{d\Omega} = \frac{e^2 a^2 \sin^2 \theta}{4\pi c^3 (1 - \beta \cos \theta)^5}, \quad \frac{dP_{\parallel, \text{observer}}}{d\Omega} = \frac{e^2 a^2 \sin^2 \theta}{4\pi c^3 (1 - \beta \cos \theta)^6}. \quad (27)$$

The dipole radiation pattern is folded forwards, as shown on the left below, with maximum intensity at angle $\theta_{\text{max}} \approx 1/2\gamma$ when $\gamma \gg 1$.



The total power radiated by the accelerated charge is,

$$P_{\parallel, \text{source}} = \int \frac{dP_{\parallel}}{d\Omega} d\Omega = \frac{e^2 a^2}{2c^3} \int_{-1}^1 \frac{1 - \cos^2 \theta}{(1 - \beta \cos \theta)^5} d\cos \theta = \frac{\gamma^6 e^2 a^2}{3c^3}, \quad (28)$$

after integrating by parts twice (or using 2.153 of [10]).

2.2.4 Acceleration Perpendicular to the Velocity

When the acceleration \mathbf{a} is perpendicular to the velocity \mathbf{v} , then $a^{*2} = \gamma^4 a^2$ according to eq. (22). Taking the acceleration \mathbf{a} to lie in the x - z plane, angle α^* is related to the polar coordinates (θ^*, ϕ^*) by,

$$\sin^2 \alpha^* = \cos^2 \theta^* + \sin^2 \theta^* \sin^2 \phi^* = \frac{(\cos \theta - \beta)^2 + (1 - \beta^2) \sin^2 \theta \sin^2 \phi}{(1 - \beta \cos \theta)^2} \quad (29)$$

The radiated power observed the lab frame follows from eqs. (14) and (18) as,

$$\begin{aligned} \frac{dP_{\perp, \text{source}}}{d\Omega} &= \frac{e^2 a^2}{4\pi c^3 (1 - \beta \cos \theta)^5} [(\cos \theta - \beta)^2 + (1 - \beta^2) \sin^2 \theta \sin^2 \phi] \\ &= \frac{e^2 a^2}{4\pi c^3 (1 - \beta \cos \theta)^5} [(1 - \beta \cos \theta)^2 - (1 - \beta^2) \sin^2 \theta \cos^2 \phi] \\ &= \frac{e^2 a^2}{4\pi c^3 (1 - \beta \cos \theta)^3} \left(1 - \frac{\sin^2 \theta \cos^2 \phi}{\gamma^2 (1 - \beta \cos \theta)^2} \right), \\ \frac{dP_{\perp, \text{observer}}}{d\Omega} &= \frac{e^2 a^2}{4\pi c^3 (1 - \beta \cos \theta)^4} \left(1 - \frac{\sin^2 \theta \cos^2 \phi}{\gamma^2 (1 - \beta \cos \theta)^2} \right). \end{aligned} \quad (30)$$

The radiation pattern is peaked in the forward direction, as shown on the right above, with the null in the pattern at $\theta \approx 1/\gamma$ for $\gamma \gg 1$.

The total power radiated by the accelerated charge is,

$$\begin{aligned} P_{\perp, \text{source}} &= \int \frac{dP_{\perp}}{d\Omega} d\Omega = \frac{e^2 a^2}{2c^3} \left(\int_{-1}^1 \frac{d \cos \theta}{(1 - \beta \cos \theta)^3} - \frac{1}{2\gamma^2} \int_{-1}^1 \frac{1 - \cos^2 \theta}{(1 - \beta \cos \theta)^5} d \cos \theta \right) \\ &= \frac{\gamma^4 e^2 a^2}{3c^3}, \end{aligned} \quad (31)$$

using 2.153 of [10] and the integral in eq. (28).

2.2.5 Invariance of the Total Power Radiated by the Source

The total power radiated by the accelerated charge can be found more quickly by expressing the radiated energy in an invariant form.

We start in the instantaneous rest frame of the charge where the integral form of the Larmor formula can be written as,

$$P_{\text{source}}^* = \frac{dU^*}{dt^*} = \frac{2e^2 a^{*2}}{3c^3} = -\frac{2e^2 a_{\nu} a^{\nu}}{3c^3}, \quad (32)$$

recalling eq. (22). The last form of eq. (22) is a Lorentz scalar, so we infer that the total power P_{source} radiated by an accelerated charge is a Lorentz scalar, and in any (inertial) frame we can write,

$$P = \frac{dU}{dt} = -\frac{2e^2 a_{\nu} a^{\nu}}{3c^3} = \frac{2\gamma^6 e^2 [a^2 - (\mathbf{v}/c \times \mathbf{a})^2]}{3c^3} = P^*. \quad (33)$$

The results (28) and (31) illustrate this general relation for the special cases that $\mathbf{a} \parallel \mathbf{v}$ and $\mathbf{a} \perp \mathbf{v}$.

If the charge has rest mass m_0 its 4-momentum is,

$$p_{\mu} = m_0 v_{\mu} = (E/c, \mathbf{p}) = (\gamma m_0 c, \gamma m_0 \mathbf{v}), \quad (34)$$

where E and \mathbf{p} and its energy and momentum. When the charge is subject to an external electromagnetic field with 4-tensor $F_{\mu\nu}$, the motion of the charge is described by,

$$\frac{dp_{\mu}}{d\tau} = m_0 a_{\mu} = f_{\mu} = e F_{\mu\nu} v^{\nu} = e\gamma(\mathbf{E} \cdot \boldsymbol{\beta}, \mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}), \quad (35)$$

so the total radiated power (33) can be written as,

$$P = -\frac{2e^2 a_{\nu} a^{\nu}}{3c^3} = \frac{2\gamma^2 e^4 [(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B})^2 - (\mathbf{E} \cdot \boldsymbol{\beta})^2]}{3m_0^2 c^3}. \quad (36)$$

For relativistic motion ($\gamma \gg 1$) in zero electric field with $\mathbf{B} \perp \mathbf{v} \approx c\hat{\mathbf{v}}$, the circular trajectory has radius $R \approx \gamma m_0 c^2 / eB$, and the radiated power is,

$$P \approx \frac{2\gamma^2 e^4 B^2}{3m_0^2 c^3} \approx \frac{2\gamma^4 c e^2}{3R^2}. \quad (37)$$

This synchrotron radiation limits the performance of circular high-energy particle accelerators.³

³See, for example, [11, 12].

2.2.6 Radiated Energy-Momentum 4-Vector

In general, an accelerated charge radiates momentum as well as energy, so we seek a 4-vector description that combines these effects.

For this, we first rewrite eq. (32) as,

$$\frac{dU^*}{dt^*} = \frac{dU^*}{d\tau} = \frac{2e^2 a^{*2}}{3c^3} = -\frac{2e^2 a_\nu a^\nu}{3c^3}, \quad (38)$$

noting that in the instantaneous rest frame the proper time interval is simply $d\tau = dt^*$. Further, the total radiated momentum \mathbf{P}^* is zero in this frame,

$$\frac{d\mathbf{P}^*}{dt^*} = \frac{d\mathbf{P}^*}{d\tau} = \int \frac{d^2\mathbf{P}^*}{dt^* d\Omega^*} d\Omega^* = \frac{1}{c} \int \hat{\mathbf{n}}^* \frac{du^*}{dt^* d\Omega^*} d\Omega^* = \frac{e^2 a^{*2}}{4\pi c^4} \int \hat{\mathbf{n}}^* \sin^2 \alpha^* d\Omega^* = 0. \quad (39)$$

If we a 4-vector for increments of the radiated energy and momentum according to,

$$dU_\mu = (dU, c d\mathbf{P}), \quad (40)$$

Then eqs. (38)-(39) suggest that we can write,

$$\frac{dU_\mu}{d\tau} = -\frac{2e^2 a_\nu a^\nu}{3c^3} b_\mu, \quad (41)$$

where b_μ is a 4-vector whose components in the $*$ frame are $(1, \mathbf{0})$. The simplest choice is that $b_\mu = v_\mu/c = \gamma(1, \mathbf{v}/c)$. Hence, it is plausible that,

$$\frac{dU_\mu}{d\tau} = -\frac{2e^2 a_\nu a^\nu}{3c^4} v_\mu. \quad (42)$$

The momentum radiated in the lab frame is then given by,

$$\frac{d\mathbf{P}}{dt} = \frac{1}{\gamma} \frac{d\mathbf{P}^*}{d\tau} = -\frac{2e^2 a_\nu a^\nu}{3c^4} \frac{\mathbf{v}}{c} = \frac{2\gamma^6 e^2 [a^2 - (\mathbf{v}/c \times \mathbf{a})^2]}{3c^5} \mathbf{v} = \frac{dU}{dt} \frac{\mathbf{v}}{c^2}. \quad (43)$$

2.2.7 Angular Distribution of Radiated Momentum

If we denote by $\hat{\mathbf{n}}$ the unit vector from the (retarded) source position to the observer in the lab frame, then the angular distribution of the momentum radiated by an accelerated charge in the lab frame is related to the angular distribution of the radiated energy by,

$$\frac{d\mathbf{P}}{dt d\Omega} = \frac{1}{c} \frac{dU}{dt d\Omega} \hat{\mathbf{n}} = \frac{1}{c} \frac{dP_{\text{source}}}{d\Omega} \hat{\mathbf{n}}. \quad (44)$$

This can be evaluated using eqs. (23) or (25), *etc.*

As an example, consider a charge e in a uniform magnetic field, for which the motion is helical in general. The acceleration $\mathbf{a} = -a \hat{\boldsymbol{\rho}}$ is perpendicular to the instantaneous velocity \mathbf{v} of the charge, and is towards the axis of the helix (to which unit vector $\hat{\boldsymbol{\rho}}$ is perpendicular). The acceleration \mathbf{a}^* in the instantaneous rest frame is also perpendicular to the axis, $\mathbf{a}^* = -\gamma^2 a \hat{\boldsymbol{\rho}}$. Hence, the radiation pattern in the instantaneous rest frame has

the form $\sin^2 \psi^*$, where angle ψ is measured with respect to $\hat{\boldsymbol{\rho}}$. This radiation pattern (in the ' frame) is not symmetric about the direction of the velocity \mathbf{v} of the charge in the lab frame.⁴ According to eq. (2), the radiation pattern in the lab frame is also not symmetric about the the direction of \mathbf{v} . Yet, according to eq. (43), the total momentum radiated in the lab frame is along \mathbf{v} (as measured by observers for which the present time in the lab is their retarded time).⁵

It is tempting to cast the expressions for the angular distributions of radiated energy and momentum into a Lorentzian form. For this, we introduce a 4-vector n_μ associated with the unit vector $\hat{\mathbf{n}}$ by defining its components in the instantaneous rest frame to be,

$$n_\mu = (0, \hat{\mathbf{n}}^*). \quad (46)$$

Then, the square of its invariant length is,

$$n_\mu n^\mu = -1. \quad (47)$$

The components of n_μ in the lab frame are,

$$n_\mu = (\gamma(\hat{\mathbf{n}}^* \cdot \boldsymbol{\beta}), \hat{\mathbf{n}}^* + (\gamma - 1)(\hat{\mathbf{n}}^* \cdot \boldsymbol{\beta})\boldsymbol{\beta}) = (n_0, \mathbf{n}). \quad (48)$$

Recalling the Larmor formula eq. (18), we can write,

$$a^{*2} \sin^2 \alpha^* = a_{\perp}^{*2} = a^{*2}(1 - \cos^2 \alpha^*) = a^{*2} - (\mathbf{a}^* \cdot \hat{\mathbf{n}}^*)^2, \quad (49)$$

where \mathbf{a}_{\perp}^ is the component of the acceleration in the instantaneous rest frame perpendicular to the direction to the observer, $\hat{\mathbf{n}}^*$. That is,*

$$\mathbf{a}_{\perp}^* = \mathbf{a}^* - (\mathbf{a}^* \cdot \hat{\mathbf{n}}^*) \hat{\mathbf{n}}^* = \mathbf{a}^* + a_\nu n^\nu \hat{\mathbf{n}}^*, \quad (50)$$

noting that $a_\nu n^\nu = -\mathbf{a}^ \cdot \hat{\mathbf{n}}^*$. This leads us to define the 4-vector,*

$$a_{\perp, \mu} = a_\mu + a_\nu n^\nu n_\mu, \quad (51)$$

⁴This can be illustrated via consideration of the ' frame that moves with velocity $-\mathbf{v}_{\parallel}$ to the lab frame, where \mathbf{v}_{\parallel} is the projection of \mathbf{v} onto the axis of the helix. In the ' frame, the motion of the charge in is a circle, and the radiation pattern (see sec. 2.2.4) is symmetric about the plane perpendicular to $\hat{\boldsymbol{\rho}}$ that contains the charge.

In particular, we consider a photon that is emitted in the forward direction in the ' frame, *i.e.*, in the direction of \mathbf{v}' . We can say that this photon is a particle with velocity $\mathbf{u}' = c \hat{\mathbf{v}}' = c \hat{\mathbf{v}}_{\perp}$.

A Lorentz transformation from the ' frame back to the lab frame takes \mathbf{v}' to \mathbf{v} and \mathbf{u}' to \mathbf{u} . The velocity transform of \mathbf{u}' is such that the angle θ_u of the velocity vector with respect to the axis of the helix is,

$$\tan \theta_u = \frac{\sqrt{1 - v_{\parallel}^2/c^2} u'_{\perp}}{u'_{\parallel} + v_{\parallel}} = \frac{\sqrt{1 - v_{\parallel}^2/c^2} c}{v_{\parallel}}, \quad (45)$$

while, noting that $\mathbf{v}'_{\perp} = \mathbf{v}_{\perp}/\sqrt{1 - v_{\parallel}^2/c^2}$, the transform for angle θ_v yields $\tan \theta_v = v_{\perp}/v_{\parallel}$. In general, $v_{\perp} \neq \sqrt{1 - v_{\parallel}^2/c^2} c$, so a photon radiated in the forward direction in the ' frame is not in the forward direction in the lab frame.

⁵This result is implied just before eq. (5) of [13], where it is noted that the radiation of energy/momentum by a charge in helical motion does not change the direction of the charge, while reducing its speed slightly.

whose invariant length squared is,

$$a_{\perp,\mu}a_{\perp}^{\mu} = a_{\mu}a^{\mu} + (a_{\nu}n^{\nu})^2 n_{\mu}n^{\mu} + 2(a_{\nu}n^{\nu})^2 = -a^{*2} + (a_{\nu}n^{\nu})^2 = -a^{*2} + (\mathbf{a}^* \cdot \hat{\mathbf{n}}^*)^2 = -a_{\perp}^{*2}. \quad (52)$$

With this notation (inspired by sec. 5.1 of [14]), we can write eq. (18) as,

$$\frac{dP^*}{d\Omega^*} = \frac{dU^*}{dt^* d\Omega^*} = \frac{e^2 a^{*2} \sin^2 \alpha^*}{4\pi c^3} = \frac{e^2 a_{\perp}^{*2}}{4\pi c^3} = -\frac{e^2 a_{\perp,\nu} a_{\perp}^{\nu}}{4\pi c^3}, \quad (53)$$

The angular distribution of radiated momentum in the $*$ frame can then be written as,

$$\frac{d\mathbf{P}^*}{dt^* d\Omega^*} = \frac{1}{c} \frac{dU^*}{dt^* d\Omega^*} \hat{\mathbf{n}}^* = -\frac{e^2 a_{\perp,\nu} a_{\perp}^{\nu}}{4\pi c^4} \hat{\mathbf{n}}^*, \quad (54)$$

It is suggestive to combine eqs. (53)-(54) into a quantity which has the appearance of a 4-vector,

$$\left(\frac{dU^*}{dt^* d\Omega^*}, c \frac{d\mathbf{P}^*}{dt^* d\Omega^*} \right) = -\frac{e^2 a_{\perp,\nu} a_{\perp}^{\nu}}{4\pi c^3} (1, \hat{\mathbf{n}}^*) = -\frac{e^2 a_{\perp,\nu} a_{\perp}^{\nu}}{4\pi c^3} \left(\frac{v_{\mu}}{c} + n_{\mu} \right) \stackrel{?}{=} \frac{dU_{\mu}}{d\tau d\Omega}. \quad (55)$$

However, this cannot be so, as can be seen in various ways. While $dU_{\mu}/d\tau$ is a 4-vector, $d\Omega$ is not a Lorentz invariant as it transforms according to eq. (12). Also, as discussed at the beginning of this section, $dU/dt d\Omega$ and $c d\mathbf{P}/dt d\Omega$ are in the ratio,

$$1 : \left(\hat{\mathbf{n}} = \frac{\mathbf{n}}{n} = \frac{\gamma[\hat{\mathbf{n}}^* + (\gamma - 1)(\hat{\mathbf{n}}^* \cdot \boldsymbol{\beta})\boldsymbol{\beta}]}{\sqrt{\gamma^2 + (\gamma^4 - 2\gamma^2 + 2\gamma - 1)(\hat{\mathbf{n}}^* \cdot \boldsymbol{\beta})^2}} \right), \quad (56)$$

whereas in the lab frame $v_0/c + n_0 = \gamma(1 + \hat{\mathbf{n}}^* \cdot \boldsymbol{\beta})$, while the 3-vector part of $v_{\mu}/c + n_{\mu}$ is,

$$\gamma\boldsymbol{\beta} + \hat{\mathbf{n}}^* + (\gamma - 1)(\hat{\mathbf{n}}^* \cdot \boldsymbol{\beta})\boldsymbol{\beta}. \quad (57)$$

Hence, this author is skeptical that eq. (5-16) of [14], i.e., eq. (55) above, holds in the lab frame, although it is correct in the instantaneous rest frame.

2.3 Acoustic Radiation

The speed c of electromagnetic radiation is the same (in vacuum) for any inertial observer, whereas the speed of acoustic radiation is different for observers with different velocities. As such, no general transformations of the angular distribution of acoustic radiation between observers with different velocities can be given for acoustic radiation.

As an example of the greater complexity of acoustic radiation compared to electromagnetic radiation, [15] quotes three different results by three different authors for the forward/backward ratio for (monopole) sound intensity for source and observer at rest, but for air moving along the direction between them. Lighthill [16], p. 476, makes the claim that the forward backward ratio is unity, independent of the speed of the air. This seems impossible for air speeds greater than the speed of sound, but it could be true for subsonic airspeeds.

For electromagnetic radiation, if the source and observer are at rest in a common inertial frame, the angular distribution of radiation in that frame is independent of the velocity of

that frame, as there is no absolute rest frame for electromagnetic waves (no electromagnetic æther). Hence, it is “obvious” that the forward/backward ratio for an isotropic electromagnetic radiator is unity, no matter what is the velocity of the frame of the source and observer. In contrast, the angular distribution (in the rest frame of the source) of sound waves emitted by a source depends, in general, on the velocity of the air in that frame; the general sense is that the angular distribution is “blown downwind”. So, it is less obvious that Lighthill’s claim is true for sound waves than for electromagnetic waves.

References

- [1] A.H. Bucherer, *Mathematische Einführung in die Elektronentheorie*, (Teubner, Leipzig, 1904), http://physics.princeton.edu/~mcdonald/examples/EM/bucherer_electrons_04.pdf
- [2] A.H. Bucherer, *Introduction Mathématique a la Théorie des Électrons*, Arch. Sci. Phys. Nat. **22**, 105 (1906), http://physics.princeton.edu/~mcdonald/examples/EM/bucherer_aspn_22_105_06.pdf
- [3] L. de la Rive, *Sur l’introduction du facteur de Doppler dans la solution des équations de la théorie des électrons*, Arch. Sci. Phys. Nat. **22**, 209, 321 (1906), http://physics.princeton.edu/~mcdonald/examples/EM/delarive_aspn_22_209_07.pdf
- [4] A.H. Bucherer, *Les Potentiels Retardés Transformés*, Arch. Sci. Phys. Nat. **23**, 129 (1906), http://physics.princeton.edu/~mcdonald/examples/EM/bucherer_aspn_23_129_07.pdf
- [5] L. de la Rive, *Sur l’introduction du facteur de Doppler dans la solution des équations de la théorie des électrons*, Arch. Sci. Phys. Nat. **23**, 433 (1907), http://physics.princeton.edu/~mcdonald/examples/EM/delarive_aspn_23_433_07.pdf
- [6] G.A. Schott, *Electromagnetic Radiation and the Mechanical Reactions Arising from it* (Cambridge U. Press, 1912), http://physics.princeton.edu/~mcdonald/examples/EM/schott_radiation_12.pdf
- [7] V.L. Ginzburg and S.I. Strovatskii, *Developments in the Theory of Synchrotron Radiation and its Reabsorption*, Ann. Rev. Astron. Astrop. **7**, 375 (1969), http://physics.princeton.edu/~mcdonald/examples/EM/ginzburg_araa_7_375_69.pdf
- [8] H. Matzner and K.T. McDonald, *Isotropic Radiators* (Apr. 8, 2003), <http://physics.princeton.edu/~mcdonald/examples/isorad.pdf>
- [9] J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, 1975), http://physics.princeton.edu/~mcdonald/examples/EM/jackson_ce2_75.pdf
- [10] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. (Academic Press, 1994), http://physics.princeton.edu/~mcdonald/examples/mechanics/gradshteyn_80.pdf
- [11] K.T. McDonald and C.G. Tully, *Maximum Energy of Circular Colliders* (Dec. 10, 2001), <http://physics.princeton.edu/~mcdonald/examples/lep.pdf>

- [12] K.T. McDonald, *The Maximal Energy Attainable in a Betatron* (Nov. 19, 2000), <http://physics.princeton.edu/~mcdonald/examples/betatron.pdf>
- [13] N.S. Kardashev, *Nonstationarity of Spectra of Young Sources of Nonthermal Radio Emission*, *Sov. Astron.* **6**, 317 (1962), http://physics.princeton.edu/~mcdonald/examples/astro/kardashev_sa_6_317_62.pdf
- [14] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, 1965; World Scientific, 2007).
- [15] C.H.E. Warren, *A Note on Moving Multipole Sources of Sound*, *J. Sound Vib.* **44**, 3 (1976), http://physics.princeton.edu/~mcdonald/examples/fluids/warren_jsv_44_3_76.pdf
- [16] J. Lighthill, *The Propagation of Sound through Moving Fluids*, *J. Sound Vib.* **24**, 471 (1972), http://physics.princeton.edu/~mcdonald/examples/fluids/lighthill_jsv_24_471_72.pdf