Self Inductance of a Solenoid with a Permanent-Magnet Core

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1 Problem

Deduce the self inductance L of a solenoidal coil of N turns of radius r and length $l \gg r$ when its core is a cylinder with permanent-magnetization density $\mathbf{M}$ parallel to the solenoid axis. Consider an oscillatory current $I(t)$ in the coil, and show that the $\mathcal{EMF}$ across the coil, due to the permanent magnet, has the character of a capacitance rather than an inductance.

2 Solution

In the limit of large l/r the magnetic field due to the permanent magnetization density $\mathbf{M} = M\,\hat{\mathbf{z}}$ is (in SI units)[1]

$$\mathbf{B}_M \approx \begin{cases} \mu_0 \mathbf{M} = \mu_0 M\,\hat{\mathbf{z}} & \text{(inside)}, \\ 0 & \text{(outside)}. \end{cases} \tag{1}$$

In the quasistatic approximation, where radiation is neglected, it seems reasonable to suppose that the magnetic field due to the current $I(t)$ is[2]

$$\mathbf{B}_I \approx \begin{cases} \mu_0 N I\,\hat{\mathbf{z}}/l & \text{(inside)}, \\ 0 & \text{(outside)}. \end{cases} \tag{2}$$

The energy stored in the magnetic field B_I, which is significant only inside the volume $\pi r^2 l$ of the cylinder, is given by

$$U_I = \int \frac{B_I^2}{2\mu_0}\,d\text{Vol} \approx \frac{1}{2}\frac{\mu_0 \pi N^2 r^2}{l} I^2. \tag{3}$$

Stored energy of the form (3) holds whenever the magnetic field is due to electrical currents that have started up from zero. This example includes permanent magnetism, which can be thought of as due to permanent "supercurrents" that do not change as the current I in the rest of the circuit varies.

We recall that the force on a small magnetic dipole $\mathbf{m}$ in an external magnetic field $\mathbf{B}_{\text{ext}}$ is[3]

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_{\text{ext}}) = -\boldsymbol{\nabla} U_m, \qquad \text{where} \qquad U_m = -\mathbf{m}\cdot\mathbf{B}_{\text{ext}}. \tag{4}$$

[1]See, for example, pp. 93-94 of [1].
[2]The direction of the z-axis is chosen such that eq. (2) holds. Then, for negative M, $\mathbf{B}_M$ is antiparallel to $\mathbf{B}_I$.
[3]See, for example, p. 87 of [2].

If the permanent magnet in the present example is held at rest with respect to the surrounding coil (with any fixed relation between the directions of the magnetization $\mathbf{M}$ and the field $\mathbf{B}_I$ of the coil), no work is done by the sum of the forces (4) on the magnetic dipoles in the magnet as the magnetic field increases from zero. Although the quantity U_m of eq. (4) changes in this process, this quantity does not correspond to a change in energy of the system. Hence, we can say that to within a constant the total energy stored in the system (other than in the power source of the electrical currents) is simply that given by eq. (3), which can be written in the familiar form

$$U_I = \frac{1}{2} L I^2, \tag{5}$$

where the self inductance L_0 of the coil is

$$L = \frac{\mu_0 \pi N^2 r^2}{l}, \tag{6}$$

which is the same as if the permanent magnet were not present.

2.1 The Permanent Magnet Can Move with Respect to the Coil

It could also be that the permanent magnet is not fixed in place with respect to the coil as the latter is energized. We suppose that the permanent magnet is small enough that it lies within the uniform field region of the coil, such that the force on the magnetic is negligible. Then, the center of mass of the magnet does not move with respect to the coil.[4]

However, if the total magnetic moment $\mathbf{M}V_M$, where V_M is the volume of the magnet, is not parallel to the magnetic field $\mathbf{B}_I$ of the coil, the permanent magnet experiences a torque,

$$\boldsymbol{\tau} = \mathbf{M}V_M \times \mathbf{B}_I = \frac{\mathbf{B}_M V_M}{\mu_0} \times \mathbf{B}_I, \tag{7}$$

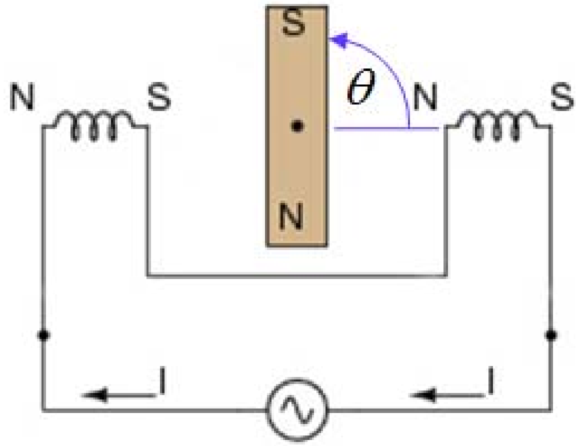
and will rotate as a consequence if the motion of the magnet is unconstrained.

If the cylindrical magnet were constrained to rotate only about its symmetry axis, which has a fixed direction (parallel to the magnetization $\mathbf{M}$), then the torque (7) has no component along the axis of rotation and does not influence the rotation of the magnet; the quantity $U_M = -\mathbf{M}V_M \cdot \mathbf{B}_I$ does not have the significance of a stored energy that can affect the self inductance of the coil.

For a more interesting case, suppose instead that the cylindrical magnet is constrained to rotate about an axis perpendicular to its symmetry axis (which is parallel to the magnetization $\mathbf{M}$), with the axis of rotation being fixed perpendicular to the symmetry axis of the coil, as sketched in the figure on the next page. Let θ be the variable angle between $\mathbf{M}$ and $\mathbf{B}_I$ and $I_{\rm mech}$ be the moment of inertia of the magnet about the its (fixed) axis of rotation.[5]

[4]If the center of mass of the magnet did move, but the magnet is small compared to the coil, the flux of magnetic field through the coil due to the permanent magnet does not change, and there would be no effect on the circuit. This is consistent with the conclusion that the center of mass of the magnet does not move.

[5]Angle $\theta = 90^\circ$ in the figure.



The rotational equation of motion of the magnet is

$$I_{\rm mech}\frac{d^2\theta}{dt^2} = \tau = -\frac{B_M V_M}{\mu_0}\frac{\mu_0 N I(t)}{l}\sin\theta = -\frac{N B_M V_M I(t)}{l}\sin\theta \tag{8}$$

If the initial angle is θ_0 and we write $\theta = \theta_0 + \vartheta$, then the equation of motion (8) can be written as

$$I_{\rm mech}\frac{d^2\vartheta}{dt^2} = -\frac{N B_M V_M I(t)}{l}(\sin\theta_0\cos\vartheta + \cos\theta_0\sin\vartheta). \tag{9}$$

In an AC circuit where the coil is in series with a resistor R and the current is $I(t) = I_0\cos\omega t$, the equation of motion of the permanent magnet becomes

$$\frac{d^2\vartheta}{dt^2} = -\frac{N B_M V_M I_0}{l I_{\rm mech}}\cos\omega t(\sin\theta_0\cos\vartheta + \cos\theta_0\sin\vartheta). \tag{10}$$

This has the oscillatory solution,

$$\cos\theta_0 = 0, \qquad \sin\theta_0 = \pm 1, \qquad \vartheta = \vartheta_0\cos\omega t, \qquad \vartheta_0 = \sin\theta_0\frac{N B_M V_M I_0}{\omega^2 l I_{\rm mech}}, \tag{11}$$

for small ϑ_0. The voltage source $V(t)$ does mechanical work on the rotating magnet at rate

$$\begin{aligned} P_{\rm mech} = \frac{d}{dt}I_{\rm mech}\dot\varphi^2 = \tau\dot\varphi &\approx -\frac{N B_M V_M I_0\cos\omega t}{l}\sin\theta_0\left(-\omega\sin\theta_0\frac{N B_M V_M I_0}{\omega^2 l I_{\rm mech}}\cos\omega t\right) \\ &= \frac{N^2 B_M^2 V_M^2}{\omega l^2 I_{\rm mech}} I_0^2\cos\omega t\sin\omega t, \end{aligned} \tag{12}$$

so the total (instantaneous) power provided by the source is, recalling eq. (5),[6]

$$\begin{aligned} P = VI &= I^2 R + \frac{dU_I}{dt} + P_{\rm mech} = I^2 R + L I\dot I + \frac{N^2 B_M^2 V_M^2}{\omega l^2 I_{\rm mech}} I_0^2\cos\omega t\sin\omega t \\ &= I\left[IR + \left(L - \frac{N^2 B_M^2 V_M^2}{\omega^2 l^2 I_{\rm mech}}\right)\dot I\right]. \end{aligned} \tag{13}$$

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[6]We might also consider the interaction magnetic-field energy, $U_{\rm int} \approx \mathbf{B}_I\cdot\mathbf{B}_M V_m/\mu_0$, which is roughly equal and opposite to the magnetic-moment energy $U_M = -\mathbf{M}V_M\cdot\mathbf{B}_I = -\mathbf{B}_I\cdot\mathbf{B}_M V_m/\mu_0 \approx -U_{\rm int}$. Since these energies largely cancel, the analysis leading to eq. (15) overestimates the effective capacitance.

Difficulties in evaluating force and energy in systems with permanent magnets are discussed, for example, in [3, 4, 5].

If we now switch to complex notation, writing $V = V_0\, e^{i\omega t}$, $I = I_0\, e^{i\omega t}$ with a I_0 complex constant, then $\dot{I} = i\omega I$ and the impedance Z of the system is

$$Z = \frac{V}{I} = R + i\omega L + \frac{N^2 B_M^2 V_M^2}{i\omega l^2 I_{\rm mech}} \equiv R + i\omega L + \frac{1}{i\omega C}\,. \tag{14}$$

We can say that the effect of the oscillating permanent magnet on the system is not so much to change the self inductance (6) of the coil,[7] but to give it an effective capacitance,

$$C = \frac{l^2 I_{\rm mech}}{N^2 B_M^2 V_M^2}\,. \tag{15}$$

The system behaves like a series *R-L-C* circuit, with resonant angular frequency

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{B_M V_M}{\sqrt{\mu_0 \pi l r^2 I_{\rm mech}}}\,, \tag{16}$$

at which frequency the magnitude of the current is maximal, with $I_0 = V_0/R$.

Electromechanical resonances have been observed in the apparatus described in [6] (private communication, David J. Jefferies). The capacitance induced by a rotating magnetic field underlies the sensor described in [7].

The magnet can also make full rotations, driven by the AC power source, in which case the system is a kind of single-phase motor, as first demonstrated by Bally [8] in 1879.[8]

References

[1] K.T. McDonald, *Electricity and Magnetism, Lecture 8*,
http://physics.princeton.edu/~mcdonald/examples/ph501/ph501lecture8.pdf

[2] K.T. McDonald, *Electricity and Magnetism, Lecture 7*,
http://physics.princeton.edu/~mcdonald/examples/ph501/ph501lecture7.pdf

[3] H.C. Lovatt and P.A. Watterson, *Energy Stored in Permanent Magnets*, IEEE Trans. Mag. **35**, 505 (1999),
http://physics.princeton.edu/~mcdonald/examples/EM/lovatt_ieeetm_35_505_99.pdf

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[7]As the permanent magnet rotates, the flux of its magnetic field through the coil varies, and an $\mathcal{EMF}$ is induced in the circuit (as pointed out by Pei-Hsun Jiang). For $\theta = \theta_0 + \vartheta$ with small ϑ, this flux is proportional to ϑ, so the $\mathcal{EMF}$ is proportional to $\dot{\vartheta}$, which is proportional to I/ω, recalling eqs. (11)-(12). While this $\mathcal{EMF}$ (reactance) is associated with magnetism, its dependence on current and frequency is that of a capacitive, rather than an inductive, reactance. It is certainly not associated with the self inductance of the circuit, since the permanent magnet is not part of the nominal electric circuit. And, since the magnet is permanent, rather than an electromagnetic, it is not to be associated with a mutual inductance in the usual sense.

[8]Variants of "flywheels" have long been used as energy storage devices in mechanical systems, and also in electromechanical systems such as motor-generator sets, although such flywheels are generally not magnets.

[5] S. Sanz *et al.*, *Evaluation of Magnetic Forces in Permanent Magnets*, IEEE Trans. Appl.. Semi. **20**, 846 (2010), http://physics.princeton.edu/~mcdonald/examples/EM/sanz_ieeetap_20_846_10.pdf

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