Hertzian Dipole Radiation via the Weizsäcker-Williams Method

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1   Problem

Use the Weizsäcker-Williams method to deduce the radiated power, and its angular distribution, emitted by an electron of charge e that undergoes oscillatory motion, $z = z_0 \cos \omega t$.

2   Solution

This problem is a continuation of an earlier note on the Weizsäcker-Williams method [1].

The basic idea of the method is that if a rapidly moving electron is sufficiently perturbed, it will emit $\alpha = e^2/\hbar c$ photons per unit bandwidth during per formation time t_0, where the latter is the time it takes for the radiation to pull one wavelength ahead of the charge. Thus, the photon number spectrum of the radiation is

$$\frac{dn}{d\omega dt_0} \approx \alpha. \tag{1}$$

Underlying the method is the fact that the electromagnetic fields of a relativistic charge are largely transverse to the direction of motion of the charge, and hence much like a free electromagnetic wave (pulse).

Here, we seek to apply this method to nonrelativistic radiation by an oscillating charge. Because the charge has a maximum velocity (relative to the speed of light c)

$$\beta = \frac{v_{\max}}{c} = \frac{z_0 \omega}{c} \tag{2}$$

much less than one, we anticipate that the radiation will not have the full strength attainable in the relativistic limit. Rather, the amplitude of the radiation will be scaled by a factor β, and so the radiated power will be scaled down by a factor β^2.[1]

Because the motion of the charge is periodic, the radiation spectrum is a line at angular frequency ω. We estimate that the rate of photon emission at frequency ω is roughly the same as that in one unit of bandwidth in the continuum version of the model.

Because the motion of the electron is bounded (with amplitude small compared to the wavelength $\lambda = 2\pi c/\omega$), the formation length is simply a wavelength, and the formation time t_0 is simply one period, $2\pi/\omega$.

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[1]A related argument was given in Sec. IIID of [1] for the case of radiation by an electron passing through a weak undulator.

Applying, the basic principle (1) of the Weizsäcker-Williams method, with the modifications described above for nonrelativistic, period motion of the charge, the number of photons radiated per unit time is

$$\frac{dN}{dt} \approx \alpha\beta^2 \frac{\omega}{2\pi} = \frac{e^2}{\hbar c}\frac{z_0^2\omega^2}{c^2}\frac{\omega}{2\pi}.$$ (3)

The radiated power is $\hbar\omega$ times this:

$$P = \hbar\omega\frac{dN}{dt} \approx \frac{(ez_0)^2\omega^4}{2\pi c^3} = \frac{p_0^2\omega^4}{2\pi c^3}.$$ (4)

where $p_0 = ez_0$ is the (maximum) dipole moment of the electron. The result (4) is about $1/2$ that of the usual expression for the time-averaged power from an oscillation dipole $p = p_0\cos\omega t$,

$$\langle P \rangle = \frac{p_0^2\omega^4}{3c^3}.$$ (5)

The factor $2\pi/3$ difference between forms (4) and (5) indicates that the effective bandwidth of the continuum that is "squeezed" into the line spectrum is only about $1/2$ unit, rather than one as assumed above.

Some care is required to determine the angular distribution of the radiation from the Weizsäcker-Williams perspective. One might be tempted to argue that the radiation will be emitted preferentially in the direction of velocity $\mathbf{v}$ of the charge, because wavelike fields that surround the charge (its cloud of "virtual photons") have $\mathbf{E} \times \mathbf{B}$ largely parallel to $\mathbf{v}$. And indeed, in the relativistic limit the radiation is emitted primarily in the $\mathbf{v}$ direction. However, the present problem is in the nonrelativistic limit, in which the average velocity of the charge is zero, so we must look elsewhere for a description of the angular distribution.

A useful argument is based on the radiation reaction. Power is emitted by the oscillating charge, so we expect a back reaction on the charge. This reaction must come from the interaction of the charge with the radiation fields. The power of the reaction force is

$$P_{\text{react}} = \mathbf{F}_{\text{react}} \cdot \mathbf{v} = e\mathbf{E}_{\text{react}} \cdot \mathbf{v} = e\mathbf{E}_{\text{rad}} \cdot \mathbf{v}.$$ (6)

To provide the required radiation reaction, the electric field $\mathbf{E}_{\text{rad}}(\theta)$ of the radiation at angle θ to the z-axis must have a large component along the direction of $\mathbf{v}$, $i.e.$, along the z axis in the present problem. Because the electric field of the radiation is transverse to the direction of propagation of the radiation, when the direction of propagation makes angle θ to the z-axis, the direction of the electric field makes the complementary angle $90° - \theta$ to the z-axis. This indicates that the electric field of radiation emitted at angle θ with respect to the z-axis varies as $\sin\theta$ (rather than $\cos\theta$ as would be too naïvely inferred from the previous paragraph).

The radiated power goes as the square of the electric field, so the angular distribution corresponding to eq. (4) is

$$\frac{dP}{d\Omega} = \frac{3p_0^2\omega^4}{8\pi^2 c^3}\sin^2\theta.$$ (7)

Or, applying the factor $2\pi/3$, we recover the usual Hertzian expression,

$$\frac{d\langle P \rangle}{d\Omega} = \frac{p_0^2\omega^4}{4\pi c^3}\sin^2\theta.$$ (8)

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References

[1] M.S. Zolotorev and K.T. McDonald, *Classical Radiation Processes in the Weizsacker-Williams Approximation* (Aug. 25, 1999), physics/0003096,
http://puhep1.princeton.edu/~mcdonald/examples/weizsacker.pdf