Another Circuit Paradox of Hering

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1 Problem

In 1911, Hering [1] performed experiments, sketched below, in which a battery-driven circuit included a conducting arm that could rotate while maintaining electrical contact with a trough of mercury.[1] The observed directions of rotation of the movable conductors are indicated in the figures below.

FIG. 8.

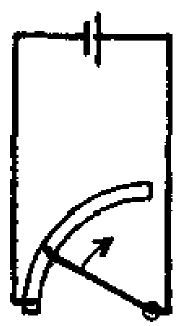
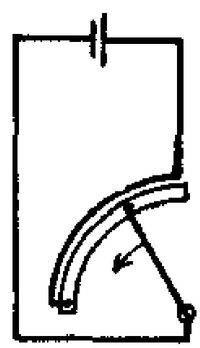


FIG. 9.



Ampère's famous analysis of the force between electrical currents (pp. 21-24 of [6]) indicated that parallel currents attract and opposite currents repel. This seems to imply that if an electric circuit could deform, it would expand, in contradiction to the results of Hering's experiments.

From this, Hering inferred that a major revision of the laws of electrodynamics is required. Can this be so?

2 Solution

Hering seemed to be among those who are skeptical of the Biot-Savart force law,

$$\mathbf{F} = \oint I\, d\mathbf{l} \times \mathbf{B}, \qquad \mathbf{B} = \frac{\mu_0}{4\pi} \oint \frac{I\, d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}\,, \tag{1}$$

which readily predicts the observed directions of rotation of the movable conductors in the above figures, noting that the largest magnetic fields on these conductors (not due to the currents in themselves) is that due to the currents in the short arc closest to the tips of the moving conductors, which fields are out of the page in both cases.

An issue for people like Hering is that the Biot-Savart force law does not satisfy Newton's $3^{\rm rd}$ law of action and reaction for isolated current elements, although it does so for closed circuits.[2]

[1] See also Fig. 4 of [2]. Other discussions of Hering by the author are in [3]-[5].

[2] Examples of related issues are cited on p. 5 of [3].

2.1 Ampère

We recall that in 1822-1823 (pp. 21-24 of [6]), Ampère formulated a force between two circuits, carrying currents I_1 and I_2, in a manner that did obey Newton's 3$^{\rm rd}$ law, written here in vector notation,

$$\mathbf{F}_{\rm on\ 1} = \oint_1 \oint_2 d^2\mathbf{F}_{\rm on\ 1}, \quad d^2\mathbf{F}_{\rm on\ 1} = \frac{\mu_0}{4\pi} I_1 I_2 [3(\hat{\mathbf{r}}\cdot d\mathbf{l}_1)(\hat{\mathbf{r}}\cdot d\mathbf{l}_2) - 2\, d\mathbf{l}_1 \cdot d\mathbf{l}_2] \frac{\hat{\mathbf{r}}}{r^2} = -d^2\mathbf{F}_{\rm on\ 2}, \quad (2)$$

where $\mathbf{r} = \mathbf{l}_1 - \mathbf{l}_2$ is the distance from a current element $I_2\, d\mathbf{l}_2$ at $\mathbf{r}_2 = \mathbf{l}_2$ to element $I_1\, d\mathbf{l}_1$ at $\mathbf{r}_1 = \mathbf{l}_1$.[3]

It is perhaps less well known that around 1825, Ampère noted, p. 214 of [8], p. 29 of [6], p. 366 of the English translation in [7], that for a closed circuit, eq. (2) can be rewritten as,[4]

$$\mathbf{F}_{\rm on\ 1} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\mathbf{l}_1\cdot\hat{\mathbf{r}})\, d\mathbf{l}_2 - (d\mathbf{l}_1\cdot d\mathbf{l}_2)\,\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\oint_1\oint_2 I_1 d\mathbf{l}_1 \times \frac{I_2\, d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}, \quad (7)$$

in vector notation (which, of course, Ampère did not use). Ampère made very little comment on this result,[5] and certainly did not factorize it into the forms now related to the Biot-Savart law(s) (1).

2.2 Electromagnetic Momentum

The reconciliation of the Biot-Savart force law with Newton's 3$^{\rm rd}$ law is that the electromagnetic field can contain momentum, such that the 3$^{\rm rd}$ law for two current elements (moving charges) should be generalized to,

$$\mathbf{F}_{\rm on\ 1} + \mathbf{F}_{\rm on\ 2} + \frac{d\mathbf{P}_{\rm EM}}{dt} = 0, \qquad \mathbf{P}_{\rm EM} = \int \epsilon_0 \mathbf{E}\times\mathbf{B}\, d{\rm Vol}. \quad (8)$$

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[3]Ampère noted the equivalents to,

$$d\mathbf{l}_1 = \frac{\partial \mathbf{r}}{\partial l_1} dl_1, \quad \mathbf{r}\cdot d\mathbf{l}_1 = \mathbf{r}\cdot\frac{\partial \mathbf{r}}{\partial l_1} dl_1 = \frac{1}{2}\frac{\partial r^2}{\partial l_1} dl_1 = r\frac{\partial r}{\partial l_1} dl_1, \quad d\mathbf{l}_2 = -\frac{\partial \mathbf{r}}{\partial l_2} dl_2, \quad \mathbf{r}\cdot d\mathbf{l}_2 = -r\frac{\partial r}{\partial l_2} dl_2, \quad (3)$$

where l_1 and l_2 measure distance along the corresponding circuits in the directions of their currents. Then,

$$d\mathbf{l}_1\cdot d\mathbf{l}_2 = -d\mathbf{l}_1\cdot\frac{\partial \mathbf{r}}{\partial l_2} dl_2 = -\frac{\partial}{\partial l_2}(\mathbf{r}\cdot d\mathbf{l}_1)\, dl_2 = -\frac{\partial}{\partial l_2}\left(r\frac{\partial r}{\partial l_1}\right) dl_1\, dl_2 = -\left(\frac{\partial r}{\partial l_1}\frac{\partial r}{\partial l_2} + r\frac{\partial^2 r}{\partial l_1 \partial l_2}\right) dl_1\, dl_2, \quad (4)$$

and eq. (2) can also be written in forms closer to those used by Ampère,

$$d^2\mathbf{F}_{\rm on\ 1} = \frac{\mu_0}{4\pi} I_1 I_2 dl_1 dl_2 \left[2r\frac{\partial^2 r}{\partial l_1\partial l_2} - \frac{\partial r}{\partial l_1}\frac{\partial r}{\partial l_2}\right]\frac{\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} 2 I_1 I_2 dl_1 dl_2 \frac{\partial^2\sqrt{r}}{\partial l_1 \partial l_2}\frac{\hat{\mathbf{r}}}{\sqrt{r}} = -d^2\mathbf{F}_{\rm on\ 2}. \quad (5)$$

[4]Note that for a fixed point 2, $d\mathbf{l}_1 = d\mathbf{r}$, and $dr = d\mathbf{r}\cdot\hat{\mathbf{r}} = d\mathbf{l}_1\cdot\hat{\mathbf{r}}$. Then, for any function $f(r)$, $df = (df/dr)\, dr = (df/dr)\, d\mathbf{l}_1\cdot\hat{\mathbf{r}}$. In particular, for $f = -1/r$, $df = d\mathbf{l}_1\cdot\hat{\mathbf{r}}/r^2$, so the first term of the first form of eq. (7) is a perfect differential with respect to $\mathbf{l}_1$. Hence, when integrating around a closed loop 1, the first term does not contribute, and it is sufficient to write (as first noted by Neumann, p. 67 of [11]),

$$\mathbf{F}_{\rm on\ 1} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1\oint_2 \frac{d\mathbf{l}_1\cdot d\mathbf{l}_2}{r^2}\hat{\mathbf{r}} = -\mathbf{F}_{\rm on\ 2}. \quad (6)$$

[5]As a consequence, the form (7) is generally attributed to Grassmann [9], as in [10], for example.

However, the abstract character of the field momentum makes some people hesitant to accept its existence, and doubts as to the consistency of our understanding of electromagnetism persist.[6]

2.2.1 Some History of Electromagnetic Field Momentum

Gaussian units are employed in this section.

Building on Faraday's electrotonic state,[7] Maxwell had a conception of electromagnetic momentum, computed as [50, 51],

$$\mathbf{P}_{\rm EM}^{(\rm Maxwell)} = \int \frac{\rho \mathbf{A}^{(C)}}{c}\, d{\rm Vol}, \tag{9}$$

where ρ is the electric charge density and $\mathbf{A}^{(C)}$ is the vector potential in the Coulomb gauge (that Maxwell used prior to the explicit recognition of gauge conditions [10]), but the form (9) seems to associate the momentum with charges rather than with fields.

In 1881, Thomson (as a 25-year-old graduate student) noted [52] that the magnetic field energy of a uniform sphere of radius a with electric charge q and velocity $v \ll c$ has the value,[8,9,10]

$$U_{\rm M} = \frac{1}{2c}\int \mathbf{J}\cdot\mathbf{A}\, d{\rm Vol} = \frac{2q^2}{15a}\frac{v^2}{c^2} = \frac{2U_{\rm E}v^2}{3c^2} \qquad \left(= \frac{q^2}{3a}\frac{v^2}{c^2} \text{ for a spherical shell}\right), \tag{10}$$

where $U_{\rm E} = \int (E^2/8\pi)\, d{\rm Vol}$. Thomson then interpreted the coefficient of $v^2/2$ in the energy $U_{\rm M}$ as mass due to the electromagnetic field,

$$m_{\rm EM} = \frac{4}{3}\frac{U_{\rm E}}{c^2}, \tag{11}$$

launching a debate as to how much of particle mass is due to fields that continues to this day.[11,12]

[6] Discussions of electromagnetism and the 3rd law in the pedagogic literature of the 1930's and 40's include [12]-[33]. Discussions by the author include [34]-[44]. The subtle topic of "hidden" momentum is discussed in [45] and references therein.

[7] Mentions by Faraday of the electrotonic state include Art. 60 of [46], Art. 1661 of [47], Arts. 1729 and 1733 of [48], and Art. 3269 of [49].

[8] Thomson's derivation involved setting $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ (*i.e.*, use of the Coulomb gauge), as favored by Maxwell (sec. 98 of [50] and sec. 617 of [51]). Fitzgerald commented on this procedure in [53], and later came to favor the Lorenz gauge [54] in which the potentials do not have instantaneous components. See also pp. 115-118 of [55], and sec. IIC of [10].

[9] The result (10) is more readily obtained on noting that for $v \ll c$ the electric field $\mathbf{E}$ of the moving charge is the instantaneous static field, while (for any constant speed) the magnetic field is $\mathbf{B} = \mathbf{v}/c \times \mathbf{E}$ (eq. (29) of [56]; see also p. 20 of [57]), such that $U_{\rm M} = \int (B^2/8\pi)\, d{\rm Vol} = (v^2/c^2)\int [E^2(1-\cos^2\theta)/8\pi]\, d{\rm Vol} = 2v^2 U_{\rm E}/3c^2$.

[10] The result (10) was verified to hold for any $v < c$ by Heaviside in 1889 [56], which analysis was subsequently noted as implying that the moving sphere is Fitzgerald-Lorentz contracted [58].

[11] In the author's view, Thomson's 1881 paper [52] marks the beginning of elementary-particle physics (at least in the English-speaking community), a topic avoided by the generations of Ampère and Maxwell (although kept alive in Germany by Weber and his followers, as reviewed, for example, in [59]). An early use of what is now called the Lorentz force law for a charged particle appears in sec. 5 of this paper (although this law was used in Weber's electrodynamics, and appears heavily disguised in sec. 599 of Maxwell's *Treatise* [51]; see also [60]).

[12] The discrepancy between eq. (11) and Einstein's $U = mc^2$ [61] is called the "4/3 problem". Some of

In 1891, Thomson noted [76] that a sheet of electric displacement $\mathbf{D}$ (parallel to the surface) which moves perpendicular to its surface with velocity $\mathbf{v}$ must be accompanied by a sheet of magnetic field $\mathbf{H} = \mathbf{v}/c \times \mathbf{D}$ according to the free-space Maxwell equation $\boldsymbol{\nabla} \times \mathbf{H} = (1/c)\,\partial\mathbf{D}/\partial t$.[13] Then, the motion of the energy density of these sheets implies there is also a momentum density, eqs. (2) and (6) of [76],

$$\mathbf{p}_{\rm EM}^{\rm (Thomson)} = \frac{\mathbf{D} \times \mathbf{H}}{4\pi c}\,. \tag{12}$$

Also in 1891, Heaviside identified the momentum of the *free ether* in sec. 26 of [71] as,[14]

$$\mathbf{p}_{\rm EM}^{\rm (Heaviside)} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c}\,. \tag{13}$$

This was a clarification of his discussion in 1886, eq. (7a) of [74], of a *magnetoelectric force* $\mathbf{D}/4\pi c \times \partial\mathbf{B}/\partial t$.[15]

In 1893, Thomson transcribed much of his 1891 paper into the beginning of *Recent Researches* [77], adding the remark (p. 9) that the momentum density (12) is closely related to the Poynting vector [87, 88],[16,17]

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}. \tag{14}$$

The form (12) was also used by Poincaré in 1900 [83], following Lorentz' convention [84] that the force on electric charge q be written $q(\mathbf{D} + \mathbf{v}/c \times \mathbf{H})$, and that the Poynting vector be $(c/4\pi)\,\mathbf{D} \times \mathbf{H}$. In 1903 Abraham [85] argued for,

$$\mathbf{p}_{\rm EM}^{\rm (Abraham)} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} = \frac{\mathbf{S}}{c^2}\,, \tag{15}$$

and in 1908 Minkowski [86] advocated the form,[18,19]

$$\mathbf{p}_{\rm EM}^{\rm (Minkowski)} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c}\,. \tag{16}$$

Thomson also gave an argument (p. 348 of [90]) that the forms (9) and (12) for field momentum are equivalent once the sources of the fields are taken into account.

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the many commentaries on this "perpetual" problem include [62]-[67].

[13]Variants of Thomson's argument were given by Heaviside in 1891, sec. 45 of [68], and much later by Feynman in sec. 18-4 of [69], where it was noted that Faraday's law, $\boldsymbol{\nabla} \times \mathbf{E} = -(1/c)\,\partial\mathbf{B}/\partial t$, combined with the Maxwell equation for $\mathbf{H}$ implies that $v = c$ in vacuum, which point seems to have been initially overlooked by Thomson, although noted by him in sec. 265 of [70].

[14]See also p. 557 of [72] and p. 495 of [73].

[15]Heaviside also mentioned this concept in 1889 on pp. 399-330 of [56].

[16]Thomson argued, in effect, that the field momentum density (12) is related by $\mathbf{p}_{\rm EM} = \mathbf{S}/c^2 = u\mathbf{v}/c^2$ [76, 77]. See also eq. (19), p. 79 of [78], and p. 6 of [79]. It turns out that the energy flow velocity defined by $\mathbf{v} = \mathbf{S}/u$ can exceed c (see, for example, sec. 2.1.4 of [38] and sec. 4.3 of [80].

[17]The idea that an energy flux vector is the product of energy density and energy flow velocity seems to be due to Umov [81], based on Euler's continuity equation [82] for mass flow, $\boldsymbol{\nabla} \cdot (\rho\mathbf{v}) = -\partial\rho/\partial t$.

[18]Minkowski, like Poynting [87], Heaviside [88] and Abraham [85], wrote the Poynting vector as $\mathbf{E} \times \mathbf{H}$. See eq. (75) of [86].

[19]For some remarks on the "perpetual" Abraham-Minkowski debate, see [89].

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