

# A Conducting Checkerboard

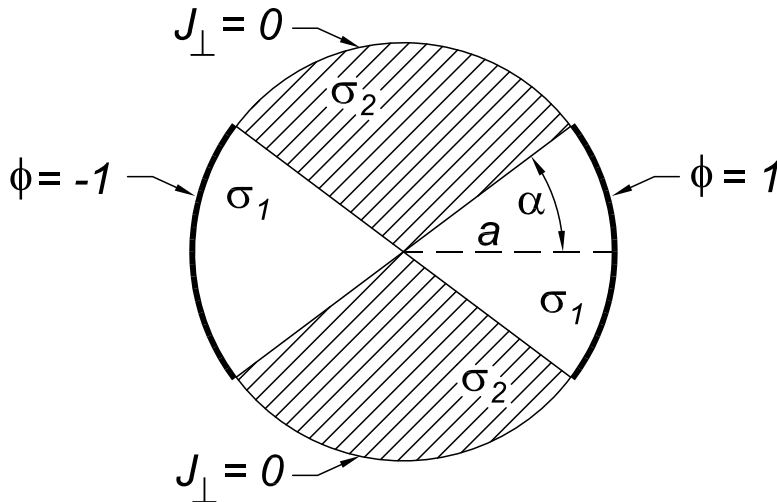
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## 1 Problem

Some biological systems consist of two “phases” of nearly square fiber bundles of differing thermal and electrical conductivities. Consider a circular region of radius  $a$  near a corner of such a system as shown below.



Phase 1, with electrical conductivity  $\sigma_1$ , occupies the “bowtie” region of angle  $\pm\alpha$ , while phase 2, with conductivity  $\sigma_2 \ll \sigma_1$ , occupies the remaining region.

Deduce the approximate form of lines of current density  $\mathbf{J}$  when a background electric field is applied along the symmetry axis of phase 1. What is the effective conductivity  $\sigma$  of the system, defined by the relation  $I = \sigma\Delta\phi$  between the total current  $I$  and the potential difference  $\Delta\phi$  across the system?

It suffices to consider the case that the boundary arc ( $r = a, |\theta| < \alpha$ ) is held at electric potential  $\phi = 1$ , while the arc ( $r = a, \pi - \alpha < |\theta| < \pi$ ) is held at electric potential  $\phi = -1$ , and no current flows across the remainder of the boundary.

Hint: When  $\sigma_2 \ll \sigma_1$ , the electric potential is well described by the leading term of a series expansion.

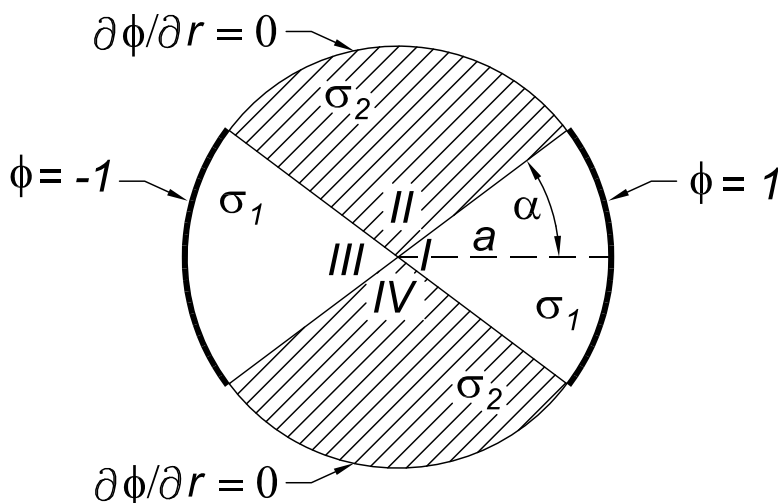
## 2 Solution

The series expansion approach is unsuccessful in treating the full problem of a “checkerboard” array of two phases if those phases meet in sharp corners as shown above. However, an analytic form for the electric potential of a two-phase (and also a four-phase) checkerboard can be obtained using conformal mapping of certain elliptic functions [1]. If the regions of one phase are completely surrounded by the other phase, rather lengthy series expansions

for the potential can be given [2]. The present problem is based on work by Grimvall [3] and Keller [4].

In the steady state, the electric field obeys  $\nabla \times \mathbf{E} = 0$ , so that  $\mathbf{E}$  can be deduced from a scalar potential  $\phi$  via  $\mathbf{E} = -\nabla\phi$ . The steady current density obeys  $\nabla \cdot \mathbf{J} = 0$ , and is related to the electric field by Ohm's law,  $\mathbf{J} = \sigma\mathbf{E}$ . Hence, within regions of uniform conductivity,  $\nabla \cdot \mathbf{E} = 0$  and  $\nabla^2\phi = 0$ . Thus, we seek solutions to Laplace's equations in the four regions of uniform conductivity, subject to the stated boundary conditions at the outer radius, as well as the matching conditions that  $\phi$ ,  $E_{\parallel}$ , and  $j_{\perp}$  are continuous at the boundaries between the regions.

We analyze this two-dimensional problem in a cylindrical coordinate system  $(r, \theta)$  with origin at the corner between the phases and  $\theta = 0$  along the radius vector that bisects the region whose potential is unity at  $r = a$ . The four regions of uniform conductivity are labeled *I*, *II*, *III* and *IV* as shown below.



Since  $\mathbf{J}_{\perp} = J_r = \sigma E_r = -\sigma\partial\phi/\partial r$  at the outer boundary, the boundary conditions at  $r = a$  can be written

$$\phi_I(r = a) = 1, \quad (1)$$

$$\frac{\partial\phi_{II}(r = a)}{\partial r} = \frac{\partial\phi_{IV}(r = a)}{\partial r} = 0, \quad (2)$$

$$\phi_{III}(r = a) = -1. \quad (3)$$

Likewise, the condition that  $j_{\perp} = j_{\theta} = \sigma E_{\theta} = -(\sigma/r)\partial\phi/\partial\theta$  is continuous at the boundaries between the regions can be written

$$\sigma_1 \frac{\partial\phi_I(\theta = \alpha)}{\partial\theta} = \sigma_2 \frac{\partial\phi_{II}(\theta = \alpha)}{\partial\theta}, \quad (4)$$

$$\sigma_1 \frac{\partial\phi_{III}(\theta = \pi - \alpha)}{\partial\theta} = \sigma_2 \frac{\partial\phi_{IV}(\theta = \pi - \alpha)}{\partial\theta}, \quad (5)$$

etc.

From the symmetry of the problem we see that

$$\phi(-\theta) = \phi(\theta), \quad (6)$$

$$\phi(\pi - \theta) = -\phi(\theta), \quad (7)$$

and in particular  $\phi(r = 0) = 0 = \phi(\theta = \pm\pi/2)$ .

We recall that two-dimensional solutions to Laplace's equations in cylindrical coordinates involve sums of products of  $r^{\pm k}$  and  $e^{\pm ik\theta}$ , where  $k$  is the separation constant that in general can take on a sequence of values. Since the potential is zero at the origin, the radial function is only  $r^k$ . The symmetry condition (6) suggests that the angular functions for region *I* be written as  $\cos k\theta$ , while the symmetry condition (7) suggests that we use  $\sin k(\pi/2 - |\theta|)$  in regions *II* and *IV* and  $\cos k(\pi - \theta)$  in region *III*. That is, we consider the series expansions

$$\phi_I = \sum A_k r^k \cos k\theta, \quad (8)$$

$$\phi_{II} = \phi_{IV} = \sum B_k r^k \sin k \left( \frac{\pi}{2} - |\theta| \right), \quad (9)$$

$$\phi_{III} = -\sum A_k r^k \cos k(\pi - \theta). \quad (10)$$

The potential must be continuous at the boundaries between the regions, which requires

$$A_k \cos k\alpha = B_k \sin k \left( \frac{\pi}{2} - \alpha \right). \quad (11)$$

The normal component of the current density is also continuous across these boundaries, so eq. (4) tells us that

$$\sigma_1 A_k \sin k\alpha = \sigma_2 B_k \cos k \left( \frac{\pi}{2} - \alpha \right). \quad (12)$$

On dividing eq. (12) by eq. (11) we find that

$$\tan k\alpha = \frac{\sigma_2}{\sigma_1} \cot k \left( \frac{\pi}{2} - \alpha \right). \quad (13)$$

There is an infinite set of solutions to this transcendental equation. When  $\sigma_2/\sigma_1 \ll 1$  we expect that only the first term in the expansions (8)-(9) will be important, and in this case we expect that both  $k\alpha$  and  $k(\pi/2 - \alpha)$  are small. Then eq. (13) can be approximated as

$$k\alpha \approx \frac{\sigma_2/\sigma_1}{k(\frac{\pi}{2} - \alpha)}, \quad (14)$$

and hence

$$k^2 \approx \frac{\sigma_2/\sigma_1}{\alpha(\frac{\pi}{2} - \alpha)} \ll 1. \quad (15)$$

Equation (11) also tells us that for small  $k\alpha$ ,

$$A_k \approx B_k k \left( \frac{\pi}{2} - \alpha \right). \quad (16)$$

Since we now approximate  $\phi_I$  by the single term  $A_k r^k \cos k\theta \approx A_k r^k$ , the boundary condition (1) at  $r = a$  implies that

$$A_k \approx \frac{1}{a^k}, \quad (17)$$

and eq. (16) then gives

$$B_k \approx \frac{1}{ka^k(\frac{\pi}{2} - \alpha)} \gg A_k. \quad (18)$$

The boundary condition (2) now becomes

$$0 = kB_k a^{k-1} \sin k \left( \frac{\pi}{2} - \theta \right) \approx \frac{k(\frac{\pi}{2} - \theta)}{a(\frac{\pi}{2} - \alpha)}, \quad (19)$$

which is approximately satisfied for small  $k$ .

So we accept the first terms of eqs. (8)-(10) as our solution, with  $k$ ,  $A_k$  and  $B_k$  given by eqs. (15), (17) and (18).

In region  $I$  the electric field is given by

$$E_r = -\frac{\partial \phi_I}{\partial r} \approx -k \frac{r^{k-1}}{a^k} \cos k\theta \approx -k \frac{r^{k-1}}{a^k}, \quad (20)$$

$$E_\theta = -\frac{1}{r} \frac{\partial \phi_I}{\partial \theta} \approx k \frac{r^{k-1}}{a^k} \sin k\theta \approx k^2 \theta \frac{r^{k-1}}{a^k}. \quad (21)$$

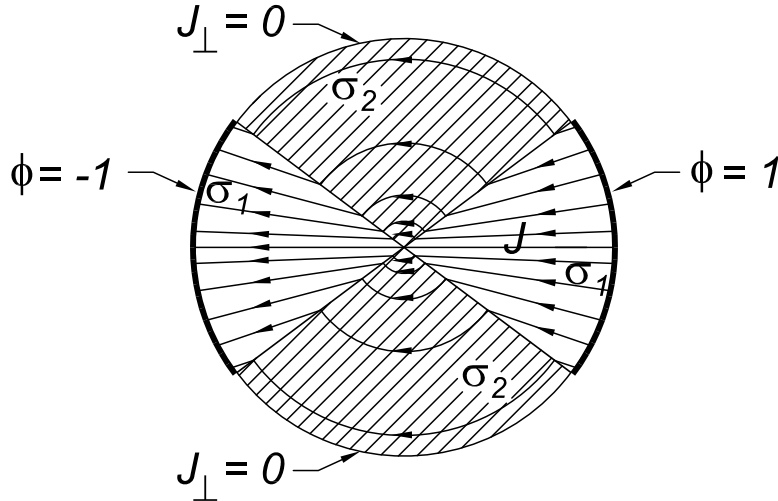
Thus, in region  $I$ ,  $E_\theta/E_r \approx k\theta \ll 1$ , so the electric field, and the current density, is nearly radial. In region  $II$  the electric field is given by

$$E_r = -\frac{\partial \phi_{II}}{\partial r} \approx -k \frac{r^{k-1}}{ka^k(\frac{\pi}{2} - \alpha)} \sin k \left( \frac{\pi}{2} - \theta \right) \approx -k \frac{r^{k-1}}{a^k} \frac{\frac{\pi}{2} - \theta}{\frac{\pi}{2} - \alpha}, \quad (22)$$

$$E_\theta = -\frac{1}{r} \frac{\partial \phi_{II}}{\partial \theta} \approx k \frac{r^{k-1}}{ka^k(\frac{\pi}{2} - \alpha)} \cos k \left( \frac{\pi}{2} - \theta \right) \approx \frac{r^{k-1}}{a^k(\frac{\pi}{2} - \alpha)}. \quad (23)$$

Thus, in region  $II$ ,  $E_r/E_\theta \approx k(\pi/2 - \theta) \ll 1$ , so the electric field, and the current density, is almost purely azimuthal.

The current density  $\mathbf{J}$  follows the lines of the electric field  $\mathbf{E}$ , and therefore behaves as sketched below:



The total current can be evaluated by integrating the current density at  $r = a$  in region  $I$ :

$$I = 2a \int_0^\alpha J_r d\theta = 2a\sigma_1 \int_0^\alpha E_r(r=a) d\theta \approx -2k\sigma_1 \int_0^\alpha d\theta = -2k\sigma_1\alpha = -2\sqrt{\frac{\sigma_1\sigma_2\alpha}{\frac{\pi}{2} - \alpha}}. \quad (24)$$

In the present problem the total potential difference  $\Delta\phi$  is  $-2$ , so the effective conductivity is

$$\sigma = \frac{I}{\Delta\phi} = \sqrt{\frac{\sigma_1\sigma_2\alpha}{\frac{\pi}{2} - \alpha}}. \quad (25)$$

For a square checkerboard,  $\alpha = \pi/4$ , and the effective conductivity is  $\sigma = \sqrt{\sigma_1\sigma_2}$ . It turns out that this result is independent of the ratio  $\sigma_2/\sigma_1$ , and holds not only for the corner region studied here but for the entire checkerboard array [5].

## References

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