Canoe ± Rock

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1 Problem

When a “rock” is thrown overboard from a canoe, into the lake in which the latter floats, does the water level in the lake rise or fall relative to an observer on the shore? Does the canoe rise or fall relative to that observer?

Consider a “rock” of density both greater or less than the density of water.

For simplicity, consider a canoe, a “rock” and a lake all of which have vertical sides.

2 Solution

A quick answer is that when the “rock” is in the floating canoe it causes water to be displaced equal to the weight of the canoe, but when the “rock” is at the bottom of the lake it displaces only its own volume, which is smaller than that of the water displaced when floating if the density of the “rock” is greater than the density of water. Hence, more water is displaced when the “rock” is in the canoe than when at the bottom of the lake, so the water level of the lake is higher when the “rock” is in the canoe.

The canoe floats higher relative to the water when it does not carry the “rock”, but the water level is higher when the canoe carries the “rock”, so it is less evident whether the canoe is higher relative to the lake shore with or without the “rock.”

2.1 Canoe + “Rock”

In more detail, the lake has area A_l, and in the absence of the canoe and “rock” the water depth h_0, such that the volume of the water is

$$V_w = h_0 A_l. \tag{1}$$

When the “rock” of weight W_r is in the canoe of area $A_c < A_l$ and weight W_c, which are floating in the lake as shown in top right of the figure below, the water depth is h_1, the canoe is submerged to depth d_1, and the bottom of the canoe is at height $H_1 = h_1 - d_1$ above the bottom of the lake,

The volume V_c of water displaced by the canoe has weight equal to $W_c + W_r$ according to Archimedes' principle,

$$W_c + W_r = V_c \rho_w g = d_1 A_c \rho_w g, \qquad V_c = \frac{W_c + W_r}{\rho_w g}, \qquad d_1 = \frac{W_c}{\rho_w A_c} + \frac{W_r}{\rho_w A_c}, \tag{2}$$

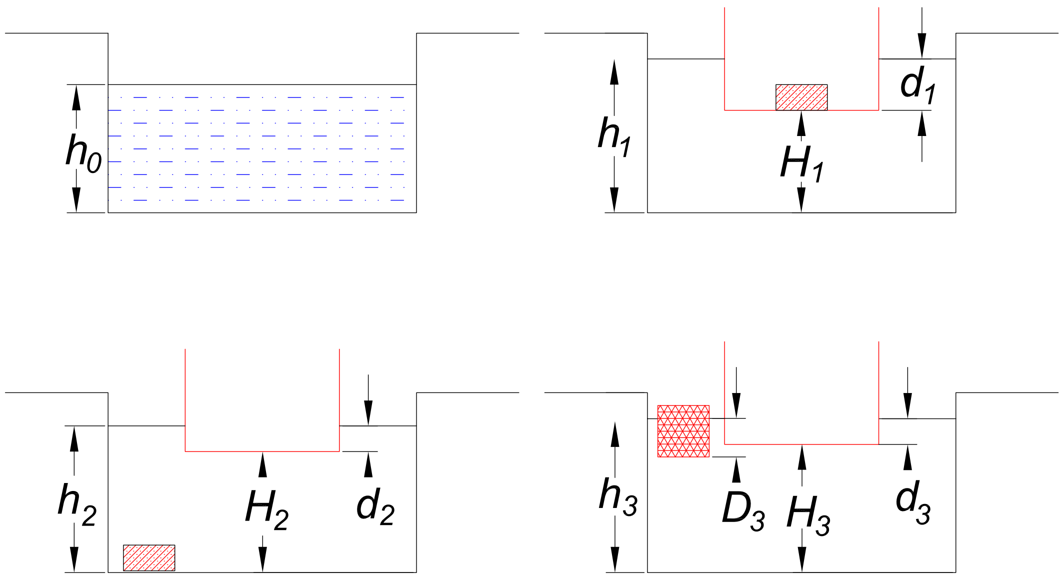
where ρ_w is the density of water and g is the acceleration due to gravity. The depth h_1 of the water in the lake is now related by

$$h_1 A_l = V_w + V_c = h_0 A_l + \frac{W_c + W_r}{\rho_w g}, \qquad h_1 = h_0 + \frac{W_c + W_r}{\rho_w g A_l} > h_0, \tag{3}$$

so the height H_1 of the canoe + "rock" above the bottom of the lake is

$$H_1 = h_1 - d_1, \qquad H_1 = h_0 + \frac{W_c + W_r}{\rho_w g}\left(\frac{1}{A_l} - \frac{1}{A_c}\right) < h_0, \tag{4}$$

since $A_c < A_l$.



2.2 "Rock" at the Bottom of the Lake

When the "rock" of density $\rho_r > \rho_w$ volume

$$V_r = \frac{W_r}{\rho_r g} \tag{5}$$

is at the bottom of the lake, the volume V_c' of water displaced by the canoe is only

$$V_c' = \frac{W_c}{\rho_w g} = d_2 A_c\,, \qquad d_2 = \frac{W_c}{\rho_w g A_c}, \tag{6}$$

and the depth h_2 of the water in the lake is now related by

$$h_2 A_l = V_w + V_c' + V_r = h_0 A_l + \frac{W_c}{\rho_w g} + \frac{W_r}{\rho_r g}\,, \qquad h_2 = h_0 + \frac{W_c}{\rho_w g A_l} + \frac{W_r}{\rho_r g A_l} > h_0. \tag{7}$$

Comparing with eq. (3), we see that

$$h_1 = h_2 + \frac{W_r}{g A_l}\left(\frac{1}{\rho_w} - \frac{1}{\rho_r}\right) = h_2 + \frac{W_r}{g A_l}\frac{\rho_r - \rho_g}{\rho_r \rho_w} > h_2 > h_0, \tag{8}$$

in agreement with the quick solution for $\rho_r > \rho_w$.

The height H_2 of the canoe above the bottom of the lake, when the "rock" is at the bottom, is given by

$$H_2 = h_2 - d_2 = h_0 + \frac{W_c}{\rho_w g A_l} + \frac{W_r}{\rho_r g A_l} - \frac{W_c}{\rho_w g A_c} = h_0 + \frac{W_r}{\rho_r g A_l} - \frac{W_c}{\rho_w g}\left(\frac{1}{A_c} - \frac{1}{A_l}\right)$$
$$= H_1 + \frac{W_r}{g}\left(\frac{1}{\rho_r A_l} - \frac{1}{\rho_w A_l} + \frac{1}{\rho_w A_c}\right) = H_1 + \frac{W_r}{\rho_r g A_c}\left[\frac{A_c}{A_l} + \frac{\rho_r}{\rho_w}\left(1 - \frac{A_c}{A_l}\right)\right] > H_1, \qquad (9)$$

since $A_c < A_l$. That is, the height of the canoe relative to the lake shore is greater when the "rock" is at the bottom of the lake than when in the canoe. A subtlety is that the bottom of the canoe is higher than h_0 in the "silly" limit of a very small lake, while for a reasonably large lake $H_2 < h_0$.

2.3 Floating "Rock"

If the density ρ_r' of the "rock" is less than the density ρ_w of water, the "rock" floats when thrown overboard, and displaces volume V_r' of water given by

$$V_r' = \frac{W_r}{\rho_w g}, \qquad (10)$$

where A_r is the (horizontal) area of the "rock" presuming it to have vertical sides. The volume V_c' of water displaced by the floating canoe and the submerged depth d_3 of the canoe are again given by eq. (6),

$$d_3 = d_2 = \frac{W_c}{\rho_w g A_c}, \qquad (11)$$

so the depth h_3 of the water in the lake is now related by

$$h_3 A_l = V_w + V_c' + V_r' = h_0 A_l + \frac{W_c}{\rho_w g} + \frac{W_r}{\rho_w g}, \quad h_3 = h_0 + \frac{W_c}{\rho_w g A_l} + \frac{W_r}{\rho_w g A_l} = h_1 > h_2 > h_0. (12)$$

Since the "rock" is floating in both cases 1 and 3, the depth of the water in the lake is the same in these two cases, if the weight of the "rock" is the same.

The height H_3 of the canoe above the bottom of the lake, when the "rock" is floating outside the canoe, is given by

$$\begin{aligned} H_3 &= h_3 - d_3 = h_0 + \frac{W_c}{\rho_w g A_l} + \frac{W_r}{\rho_w g A_l} - \frac{W_c}{\rho_w g A_c} = h_0 + \frac{W_r}{\rho_w g A_l} - \frac{W_c}{\rho_w g}\left(\frac{1}{A_c} - \frac{1}{A_l}\right), \\ &= H_1 + \frac{W_r}{\rho_w g A_c} > H_1, \\ &= H_2 + \frac{W_r}{g A_l}\left(\frac{1}{\rho_w} - \frac{1}{\rho_r}\right) > H_2 > H_1, \end{aligned} \qquad (13)$$

where in the last line $\rho_r > \rho_w$ is the density of the "rock" that sinks in case 2, rather than the density ρ_r' of the floating "rock" in case 3. For large lakes, $H_3 < h_0$, but for very small lakes it can be that $H_3 > h_0$. *In the example illustrated on p. 1, $H_1 = h_0$ and $H_3 > h_0$.*