

# Birkeland, Darboux and Poincaré: Motion of an Electric Charge in the Field of a Magnetic Pole

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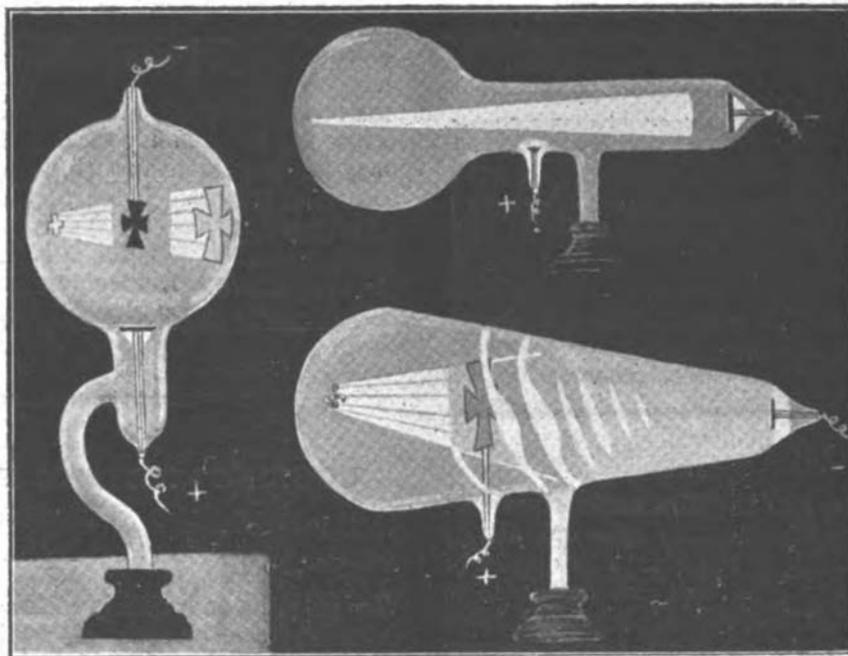
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## 1 Problem

In 1896, Birkeland reported [1, 2, 3, 4] studies of a Crookes tube [5] when a strong, straight electromagnet was placed outside and to the left, as in the figures below.

**FIG. 2.**

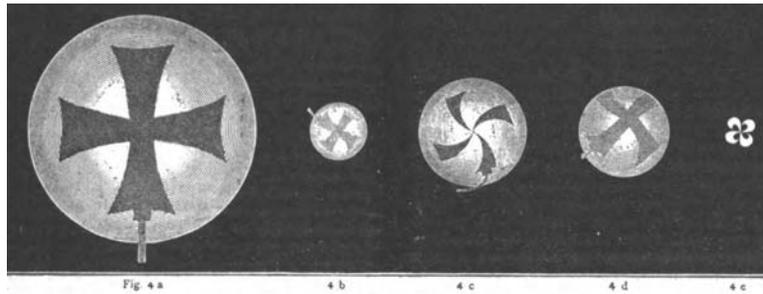


**FIG. 1.**

**FIG. 3A.**



In Fig. 1, the cathode was the cross in the center of the tube, from which cathode rays emanated both to the left and to the right, following lines of the magnetic field (which diverges from the magnet pole to the left of the tube). In Fig. 2 the cathode was at the far right of the tube, and the cathode rays simply converged towards the magnet pole at the left. In Fig. 3a the anode was the cross near the center of the tube, with the cathode at the right end; only those cathode rays that didn't intercept the anode proceeded past it to the left, converging towards the magnet pole. Further, in experiments with the configuration of Fig. 3a, the size of the shadow on the left face of the tube due the anode cross varied in an oscillatory manner with the distance between the magnet and the tube, illustrated below where Fig. 4a corresponds to a 20-cm gap between the magnet pole and the tube, while the magnet pole touched the tube in Fig. 4e.



Birkeland also remarked “When a cylindrical discharge tube is placed in a uniform magnetic field between the poles of large horse-shoe magnet, there is no convergence of the cathode rays to a focus; it is therefore, possible that this effect in the experiments described above is due to the convergence of the lines of force in the field at the end of a straight magnet.”

Experiments similar to those of Birkeland were being conducted elsewhere in 1896. Indeed, the lower figure on p. 1 is from Swinton [6], and corresponds to fig. 2 above of Birkeland, in which the pole of a magnet located far from the cathode resulted in a conical beam converging on the pole.

The nature of cathode rays was not understood in 1896, which were “discovered” to be electrons by J.J. Thomson in 1897 [7, 8] (in experiments with Crookes tubes and magnets).

Approximate the magnetic field inside the Crookes tube as due to a single magnetic pole of strength  $p$ , and discuss the motion of an electron (in vacuum) subject to this magnetic field.

*Birkeland’s scientific efforts are honored on the 200-kroner Norwegian banknote.*



## 2 Poincaré's Solution

This section follows a rather brilliant paper by Poincaré (1896) [9] (under whom Birkeland had studied in 1892), which seems to be based on a much earlier paper by Darboux (1978) [10],<sup>1</sup> into which it is possible to read more than was perhaps understood at the time.

### 2.1 General Relations

Taking the magnetic pole to be at the origin, its magnetic field is (in Gaussian units),

$$\mathbf{B} = \frac{p\mathbf{r}}{r^3}. \quad (1)$$

Using the Lorentz force law (1892) [12, 13] for a particle of electric charge  $e$  and mass  $m$ , its equation of motion is, with  $c$  being the speed of light in vacuum,

$$m \frac{d^2\mathbf{r}}{dt^2} = \frac{e}{c} \frac{d\mathbf{r}}{dt} \times \mathbf{B} = \frac{ep}{cr^3} \frac{d\mathbf{r}}{dt} \times \mathbf{r} \quad (\text{Poincaré's 1st equation, Darboux's 1st equation}). \quad (2)$$

Taking the scalar product of eq. (1) with  $d\mathbf{r}/dt$  (see also pp. 22-24 of [14]), and then also with  $\mathbf{r}$ , we have that,

$$\begin{aligned} 0 &= \frac{1}{2} \frac{d\mathbf{r}}{dt} \cdot \frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt} \left( \frac{d\mathbf{r}}{dt} \right)^2, & \left( \frac{d\mathbf{r}}{dt} \right)^2 &= v_0^2, & 0 &= 2\mathbf{r} \cdot \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2r^2}{dt^2} - 2 \left( \frac{d\mathbf{r}}{dt} \right)^2, \\ \frac{d^2r^2}{dt^2} &= 2v_0^2, & r^2 &= v_0^2 t^2 + 2Bt + r_{\min}^2 & (\text{Poincaré's 2nd equation, Darboux's 3rd}). \end{aligned} \quad (3)$$

where  $r_{\min}$ ,  $B$  and  $v_0$  are constants, with  $r_{\min}$  being the distance of closest approach between the charge and the pole, and  $v_0$  being the constant magnitude of the charge's velocity.<sup>2</sup>

Next, we take the vector cross product of eq. (2) with  $\mathbf{r}$ ,

$$\begin{aligned} \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt} \left( \mathbf{r} \times \frac{d\mathbf{r}}{dt} \right) = \frac{ep}{mcr^3} \mathbf{r} \times \left( \frac{d\mathbf{r}}{dt} \times \mathbf{r} \right) = \frac{ep}{mcr} \frac{d\mathbf{r}}{dt} - \frac{ep}{2mcr^3} \frac{dr^2}{dt} \mathbf{r} = \frac{ep}{mc} \frac{d\mathbf{r}}{dt}, \\ \mathbf{r} \times m \frac{d\mathbf{r}}{dt} &= \frac{ep\mathbf{r}}{c} + \mathbf{J} \quad (\text{Poincaré's 3rd equation, Darboux's 4th equation}), \end{aligned} \quad (4)$$

where  $\mathbf{J}$  is a constant vector that can be determined from the initial conditions. Then, taking the scalar product of eq. (4) with  $\mathbf{r}$ , we find that,

$$\hat{\mathbf{r}} \cdot \mathbf{J} = -\frac{ep}{c} \quad (\text{Poincaré's 4th equation, Darboux's 5th equation}), \quad (5)$$

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<sup>1</sup>Darboux was 8 years older than Poincaré, and survived him to write a scientific biography (eulogy) of the latter [11].

<sup>2</sup>That the velocity  $d\mathbf{r}/dt$  of an electric charge in a magnetic field remains constant in magnitude (such that its kinetic energy is constant) is a now-familiar feature of the Lorentz force law, but was perhaps not well known in 1896. Since the velocity  $v_0$  of the electric charge is constant, its "relativistic mass"  $m = m_0/\sqrt{1 - v_0^2/c^2}$  is constant, where  $m_0$  is the rest mass of the electron, and Poincaré's analysis actually holds for relativistic motion of the electron, assuming that the magnetic pole remains fixed at all times.

which implies that the trajectory (*i.e.*, velocity  $d\mathbf{r}/dt$ ) of the electric charge lies on the surface of a cone whose axis is parallel to the constant vector  $\mathbf{J}$ . If the velocity  $d\mathbf{r}/dt$  of the electric charge points toward the magnetic charge the motion is in a straight line, passing through the magnetic charge; hence the magnetic charge is at the apex of the cone and lines of the magnetic field are generatrices of the cone. Then, eq. (2) tells us that the acceleration  $d^2\mathbf{r}/dt^2$  of the electric charge is perpendicular to the surface of the cone, such that the motion of the charge is on a straight line (geodesic) on the surface of the cone.<sup>3</sup>

Suppose the electric charge originated at  $\mathbf{r}_0 = (x_0, 0, z_0 \gg x_0)$  with initial velocity  $d\mathbf{r}_0/dt = -v_0 \hat{\mathbf{z}}$ . That is, we choose the directions of  $x$ - and  $z$ -axes accordingly.<sup>4</sup> Then, the constant vector  $\mathbf{J}$  is,

$$\mathbf{J} = -mv_0 x_0 \hat{\mathbf{y}} - \frac{ep}{c} \hat{\mathbf{z}} \quad (\text{Poincaré's 6th equation}). \quad (6)$$

Considering the vector  $\mathbf{J}$  to be the axis of the cone on whose surface the electric charge travels, we take  $\mathbf{J}$  as pointing away from the magnetic pole at the origin. Then,

$$\mathbf{J} \cdot \hat{\mathbf{r}}_0 = -J \cos \theta = \frac{J_z z_0}{\sqrt{x_0^2 + z_0^2}} \approx J_z, \quad \cos \theta \approx \frac{|J_z|}{J}, \quad \sin \theta \approx \frac{|J_y|}{J}, \quad (7)$$

which is the general case of Poincaré's 7th equation.

### 2.1.1 Angular Velocity of Rotation about the Constant Vector $\mathbf{J}$

Poincaré did not carry the general discussion further in [9], but he later gave indications on pp. 22-26 of [14] of how this could be continued.

The (constant) vector  $\mathbf{J}$  can be decomposed into components parallel and perpendicular to  $\hat{\mathbf{r}}$  according to,

$$\mathbf{J} = (\mathbf{J} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} + \hat{\mathbf{r}} \times (\mathbf{J} \times \hat{\mathbf{r}}) = -\frac{ep}{c}\hat{\mathbf{r}} + \hat{\mathbf{r}} \times (\mathbf{J} \times \hat{\mathbf{r}}), \quad (8)$$

using eq. (5). Then, since,

$$\mathbf{r} \times \frac{d\mathbf{r}}{dt} = r^2 \hat{\mathbf{r}} \times \frac{d\hat{\mathbf{r}}}{dt}, \quad (9)$$

eq. (4) can be rewritten as,

$$\hat{\mathbf{r}} \times \left( \frac{d\hat{\mathbf{r}}}{dt} - \frac{\mathbf{J}}{mr^2} \times \hat{\mathbf{r}} \right) = 0, \quad (10)$$

Now, neither  $d\hat{\mathbf{r}}/dt$  nor  $\mathbf{J} \times \hat{\mathbf{r}}$  have components along  $\hat{\mathbf{r}}$ , so we learn that,

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{\mathbf{J}}{mr^2} \times \hat{\mathbf{r}} \equiv \boldsymbol{\Omega} \times \hat{\mathbf{r}}, \quad (11)$$

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<sup>3</sup>That is, if the cone were unrolled into a sector on a plane, the trajectory would consist of straight line segments on this planar sector. See the figure on p. 7.

<sup>4</sup>The distance  $x_0$  is called the **impact parameter** in the language of scattering theory.

where,

$$\boldsymbol{\Omega} = \frac{\mathbf{J}}{mr^2} = \frac{\mathbf{J}}{m(v_0^2 t^2 + Bt + r_{\min}^2)}, \quad (12)$$

is the angular velocity of rotation of the unit vector  $\hat{\mathbf{r}}$  about the axis  $\mathbf{J}$  of the cone on which the electric charge moves.

We can use eq. (11) to deduce a relation for the distance  $r_{\min}$  of closest approach of the electric charge to the magnetic pole.

Recall that in the first line of eq. (4) we found,

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{1}{r} \frac{d\mathbf{r}}{dt} - \frac{\hat{\mathbf{r}}}{r} \frac{dr}{dt}. \quad (13)$$

Then, when  $r = r_{\min}$  we have that  $dr/dt = 0$  and  $\hat{\mathbf{r}}$  lies along a generatrix of the cone, such that,

$$\frac{1}{r_{\min}} \frac{d\mathbf{r}}{dt} = \frac{d\hat{\mathbf{r}}}{dt} = \boldsymbol{\Omega} \times \hat{\mathbf{r}}, \quad \frac{v_0}{r_{\min}} = \Omega \sin \theta = \frac{J \sin \theta}{mr_{\min}^2}, \quad (14)$$

Hence, recalling eq. (7),<sup>5</sup>

$$r_{\min} = \frac{J \sin \theta}{mv_0} = \frac{|J_y|}{mv_0} = x_0 \quad (\text{Poincaré's 8th equation}). \quad (15)$$

Furthermore, we can define the time  $t = 0$  to be moment when the electric charge is closest to the magnetic pole, such that eq. (3) can be written as,

$$r^2 = v_0^2 t^2 + x_0^2. \quad (16)$$

Then, we can integrate eq. (12) to find the azimuth  $\Delta\phi$  of the electric charge on the cone, relative to the azimuth of  $\mathbf{r}_{\min}$  = the point of closest approach between the charge and the pole,

$$\Delta\phi = \int_0^t \Omega dt = \int_0^t \frac{J}{m(v_0^2 t^2 + x_0^2)} dt = \frac{J}{mv_0 x_0} \tan^{-1} \frac{v_0 t}{x_0}, \quad v_0 t = x_0 \tan \frac{mv_0 x_0 \Delta\phi}{J}. \quad (17)$$

## 2.2 Considerations of Birkeland's Experiment

In his analysis of Birkeland's experiment, Poincaré appears to have assumed tacitly that  $mv_0 x_0 \ll ep/c$  for a strong magnetic pole  $p$  (see sec. 3.16 for the case that  $mv_0 x_0 \gg ep/c$ ). Note that in this approximation,

$$J \approx \frac{ep}{c}, \quad \text{and} \quad \theta \approx \frac{mv_0 x_0}{J} \ll 1 \quad \left( mv_0 x_0 \ll \frac{ep}{c} \right). \quad (18)$$

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<sup>5</sup>If  $x_0 = 0$  then the initial velocity of the electric charge is along a field line and the electron moves along this line, colliding with the magnetic pole. In this case the minimum distance of approach of the charge to the pole is  $x_0 = 0$ .

It turns out that for extremely small but nonzero  $x_0$  a charged particle with spin (such as an electron or proton) also collides with the pole in a classical analysis, as discussed in sec. 3.21.

In this section we suppose that the electric charge is an electron.

In this case, the half angle of the cone is greater than  $90^\circ$ , so it is more convenient to consider its complement  $\theta < 90^\circ$ , which is related by,

$$\sin \theta = \frac{|J_y|}{J} = \frac{|J_y|}{\sqrt{J_y^2 + J_z^2}} \approx \frac{|J_y|}{|J_z|} = \frac{mcv_0x_0}{ep} \approx \theta \quad (\text{Poincaré's 7th equation}). \quad (19)$$

Although angle  $\theta$  is small, it is much larger than the angle  $\tan^{-1}(x_0/z_0)$  between the  $z$ -axis and the initial vector  $\mathbf{r}_0$ , so it is a good approximation in (most of) the rest of the discussion to suppose that  $\mathbf{r}_0$  is essentially along the  $z$ -axis. Then, the  $z$ -axis is (approximately) a generatrix of the cone, and as the electron moves on this cone it can/will cross (or at least come extremely close to) the  $z$ -axis at various places. Extending the argument to a ring source of electrons with radius  $x_0$  in the plane  $z_0$  (with  $z_0 \gg x_0$ ), we see that these  $z$ -intercepts are foci of the electron beam. Of course, the location of the foci in  $z$  depends on the value of  $x_0$ , so the experimental evidence of such foci is somewhat dependent on the details of the cathode disk/ring (and of the anode cross that blocks some of the cathode rays).

To apply eq. (17) here, we first need to know the azimuth of the point of closest approach between the electron and the pole. For this, we use eq. (17) for time  $t_0$  when the charge is as distance  $r_0 = \sqrt{v_0^2 t_0^2 + x_0^2} \approx v_0 t_0 \approx z_0$  from the pole,

$$r_0 \approx v_0 t_0 = x_0 \tan \frac{mv_0 x_0 \Delta\phi_0}{J} \approx z_0, \quad \tan \frac{mv_0 x_0 \Delta\phi_0}{J} \approx \frac{z_0}{x_0} \gg 1, \quad (20)$$

which requires  $\Delta\phi_0 \approx \pi/2$ . Hence, the trajectory of the electron crosses the  $z$ -axis whenever its azimuth relative to that of the point of closest approach to the pole is  $\Delta\phi = 2n\pi - \pi/2$ . Labeling the times at which this occurs as  $t_n$ , we have that,

$$v_0 t_n = x_0 \tan \frac{(2n\pi - \pi/2) mv_0 x_0}{J} \approx x_0 \tan[(2n\pi - \pi/2) \sin \theta], \quad (21)$$

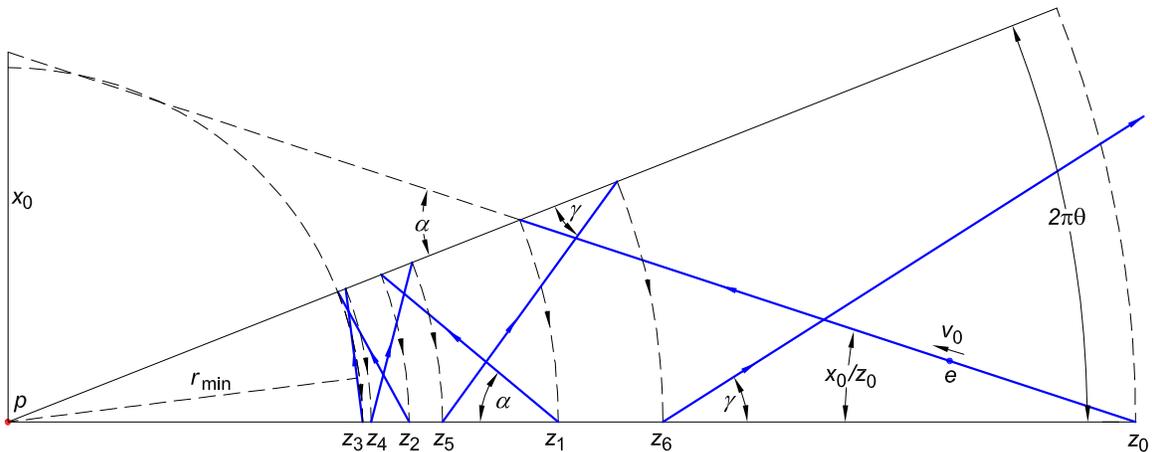
$$r_n^2 = v_0^2 t_n^2 + x_0^2 = x_0^2 (1 + \tan^2[(2n\pi - \pi/2) \sin \theta]) = \frac{x_0^2}{\cos^2[(2n\pi - \pi/2) \sin \theta]} \quad (22)$$

$$r_n = \frac{x_0}{\cos[(2n\pi - \pi/2) \sin \theta]} \approx \frac{x_0}{\cos(2n\pi \sin \theta)} \quad (\text{Poincaré's 9th equation}). \quad (23)$$

This is Poincaré's 9th equation [9], except with  $\cos(2n\pi \sin \theta)$  rather than  $\sin(2n\pi \sin \theta)$ .

An argument that emphasizes the focus at the largest distance from the magnetic pole is based on unrolling the cone of half angle  $\theta$  onto a planar sector of angle  $2\pi\theta$ , as shown below.

We recall that an electron which starts at  $z_0$  with velocity  $v_0$  in the  $-z$  direction does not have velocity parallel to the generatrix  $\mathbf{r}_0$  of the cone, but makes tiny angle  $x_0/z_0$  to this generatrix. So, when we approximate this generatrix as the  $z$ -axis we should consider that the initial velocity of the electron makes angle  $x_0/z_0$  to the generatrix/ $z$ -axis.



As noted earlier, the trajectory of the electron is a geodesic on the cone, so a straight line on the unfolded planar sector. This straight line intercepts the upper boundary of the cone at distance  $z_1$  from the origin = location of the magnetic pole  $p$ . Since this upper boundary is equivalent to the  $z$ -axis, the first intercept/focus is at  $z_1$ , which is related by,

$$2\pi\theta z_1 \approx (z_0 - z_1) \frac{x_0}{z_0}, \quad z_1 = \frac{z_0}{1 + 2\pi\theta z_0/x_0} \approx \frac{z_0}{1 + 2\pi m v_0 z_0/J}. \quad (24)$$

This argument then predicts a series of foci (crossings of the  $z$ -axis by the electron's trajectory) at decreasing distance from the magnetic pole related by,

$$z_n = z_{n-1} \frac{2(n-1)\pi\theta + x_0/z_0}{2n\pi\theta + x_0/z_0}, \quad (25)$$

so long as  $2n\pi\theta + x_0/z_0 < 90^\circ$ , after which the successive foci are farther from the pole.

The construction above also illustrates that  $r_{\min} = x_0$ , and that  $\phi_{\min} \approx \pi/2$  when  $x_0 \ll z_0$ .

### 2.3 What Poincaré (and Darboux) Didn't Say

Poincaré's discussion of Birkeland's experiments assumed the validity of the emerging "electron theory" of Lorentz (and others, particularly J.J. Thomson and O. Heaviside), but did not claim that Birkeland's experiments, or his own analysis, established this theory as the then-best description of Nature. Only in the following year (1897), when J.J. Thomson presented evidence that cathode rays consist of negatively charged particles with a unique charge-to-mass ratio [7, 8], did the "electron theory" become generally accepted.

We have written eq. (4) such that the lefthand side is "obviously" the mechanical angular momentum of the electron, which suggests that the constant vector  $\mathbf{J}$  is the total (conserved) angular momentum of the system. In this interpretation, the quantity  $-ep\hat{\mathbf{r}}/c$ , where the unit vector  $\hat{\mathbf{r}}$  points from the magnetic pole  $p$  to the electric charge  $e$ , would be the angular momentum associated with the electromagnetic fields of the system (whether or not the electric charge is at rest). In particular, if the electric charge is at rest, such that the mechanical angular momentum of the is zero, eq. (4) suggests that the total angular momentum of the system is nonzero.<sup>6</sup>

<sup>6</sup>This phenomenon was later popularized as the Feynman disk paradox [15, 16] (with the magnetic field due to a small, straight electromagnet, as in Birkeland's experiments).

However, Poincaré indicated no awareness of this physical interpretation in his 1896 paper [9] (nor did Darboux in 1878 [10]). The notion that the electromagnetic field of an apparently static system might contain nonzero angular momentum, ostensibly a dynamical quantity, was effectively “hidden” even from such a brilliant commentator as Poincaré.

Furthermore, there is no indication that Poincaré considered a nominally static electromechanical system might contain nonzero electromagnetic field momentum, which possibility has since been characterized as “hidden” momentum [17].

### 2.3.1 Poincaré on Action and Reaction Involving the Lorentz Force

Like Ampère in the 1820’s [18, 19], Poincaré was bothered in the 1890’s by the apparent violation of Newton’s 3rd law by the Lorentz force, which may be why he hesitated to give a full endorsement of the new “electron theor”. Perhaps his first statement of this is on p. 294 of [20].<sup>7</sup> That he had still not resolved this issue in 1899 is indicated by sec. 352 (p. 453) of [23]. In 1900, Poincaré provided his own deduction of electromagnetic field momentum, stating its density to be,

$$\mathbf{p}_{\text{EM}}^{(\text{Poincare})} = \frac{\mathbf{D} \times \mathbf{H}}{4\pi c} = \frac{\mathbf{S}}{c^2}, \quad (26)$$

where  $\mathbf{S}$  is the Poynting vector [25] in the notation of Lorentz [12, 13], who wrote that the force on electric charge  $e$  is  $e(\mathbf{D} + \mathbf{v}/c \times \mathbf{H})$ . Poincaré’s form (26) is the same as eq. (33) of J.J. Thomson [26].

Then, Newton’s law reads,

$$\mathbf{F}_{\text{total}} = \frac{d\mathbf{P}_{\text{total}}}{dt}, \quad \mathbf{P}_{\text{total}} = \mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{EM}} = \sum m_i \mathbf{v}_i + \int \mathbf{p}_{\text{EM}} d\text{Vol}, \quad (27)$$

and in a broad sense Newton’s principle of action and reaction is restored.<sup>8</sup>

Poincaré followed his later discussion of the motion of an electric charge in the field of a magnetic pole, pp. 22-26 of [14], by a review (pp. 26-32) of action and reaction in Lorentz’ electron theory. On p. 30 he described the electromagnetic field momentum  $\mathbf{P}_{\text{EM}} = \int \mathbf{p}_{\text{EM}} d\text{Vol}$  as the *quantité de mouvement électro-magnétique*. On p. 32 he noted that the Poynting vector has dimensions of energy density times velocity, which suggested that the momentum density (26) of the electromagnetic is associated with a flow of energy (*mouvement électro-magnétique*).

He seemed uncomfortable with this interpretation, and ended his discussion with the paragraph: *C’est là une fiction pure; car une partie de l’énergie peut disparaître en se transformant en chaleur par exemple; tandis que dans les quantités de mouvement la masse mécanique reste invariable*. This concern is a precursor to the ongoing debates on the theme of “hidden” momentum [17] mentioned earlier.<sup>9</sup>

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<sup>7</sup>On that same page, Poincaré refers to work by J.J. Thomson [21, 22] which includes the first statement, eq. (33), of the concept of electromagnetic field momentum, but Poincaré apparently did not realize this.

<sup>8</sup>Poincaré’s efforts on this topic are reviewed in [27]. See [28] for an example of how this issue is resolved.

<sup>9</sup>Field energy is transformed to heat only in cases of resistive media. See [29], particularly secs. 2.6.2 and 2.7, for a “static” example involving a nonconducting medium, in which the flow of field energy (described by the Poynting vector) must be accompanied by a “hidden” flow of “mechanical” energy (and “hidden” momentum) in the medium.

Poincaré seems not to have understood that his 3rd equation, (4),

$$\mathbf{J} = \mathbf{r} \times m \frac{d\mathbf{r}}{dt} - \frac{ep}{c} \hat{\mathbf{r}} = \mathbf{L}_{\text{mech}} + \mathbf{L}_{\text{EM}}, \quad (28)$$

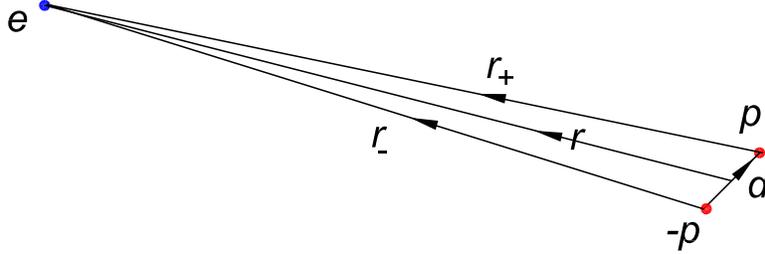
is a related illustration that conservation of angular momentum involves action and reaction between mechanical angular momentum and electromagnetic-field angular momentum.

### 2.3.2 Motion of an Electron in the Field of a Magnetic Dipole at Rest

The extrapolation of Birkeland's experiment to the case of an electron moving in the field of a magnetic dipole at rest was mentioned by Poincaré on p. 24 of [14]. No analysis was given, but it was noted that this case is relevant to the *aurora borealis* (a major interest of Birkeland after 1896).

For a "point" magnetic dipole  $\mathbf{m} = p\mathbf{d}$  that consists of magnetic poles  $\pm p$  separated by tiny distance  $\mathbf{d}$  such that  $m = pd$  is finite, an immediate extension of the analysis of sec. 2.1 is that the system of an electron plus the magnetic dipole, with the latter at rest, has the constant vector,

$$\mathbf{J} = \mathbf{r} \times m \frac{d\mathbf{r}}{dt} - \frac{ep}{c} \hat{\mathbf{r}}_+ + \frac{ep}{c} \hat{\mathbf{r}}_- = \mathbf{r} \times m \frac{d\mathbf{r}}{dt} - \frac{ep}{c} \left( \frac{\mathbf{r}_+}{r_+} - \frac{\mathbf{r}_-}{r_-} \right). \quad (29)$$



Referring to the figure above, we see that  $\mathbf{r}_{\pm} = r \hat{\mathbf{r}} \mp \mathbf{d}/2$  and  $r_{\pm} \approx r(1 \mp \hat{\mathbf{r}} \cdot \mathbf{d}/2)$ , such that,

$$\begin{aligned} \mathbf{J} - \mathbf{r} \times m \frac{d\mathbf{r}}{dt} &= -\frac{ep}{c} \left( -\frac{\mathbf{d}}{r} + \frac{(\hat{\mathbf{r}} \cdot \mathbf{d}) \hat{\mathbf{r}}}{r} \right) = \frac{e}{cr} [\mathbf{m} - (\hat{\mathbf{r}} \cdot \mathbf{m}) \hat{\mathbf{r}}] = \frac{e}{cr^2} \mathbf{r} \times (\mathbf{m} \times \hat{\mathbf{r}}) \\ &= \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{m}}{c} \end{aligned} \quad (30)$$

where  $\mathbf{E} = -e\hat{\mathbf{r}}/r^2$  is the electric field of the charge  $e$  at the magnetic dipole  $\mathbf{m}$ . This suggests that the electromagnetic field angular momentum of an electric charge together with a magnetic dipole is,

$$\mathbf{L}_{\text{EM}} = \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{m}}{c} \quad (31)$$

and further that the electromagnetic field momentum of this system is,

$$\mathbf{P}_{\text{EM}} = \frac{\mathbf{E} \times \mathbf{m}}{c}. \quad (32)$$

Both of these quantities can be nonzero when the electron (and magnetic dipole) are at rest. The latter result is particularly disconcerting, as we expect the total momentum of a system “at rest” to be zero. However, Poincaré appears not to have commented on this issue (unless his remark, *C’est là une fiction pure*, mentioned above refers to the notion of electromagnetic momentum and angular momentum for “static” systems).

### 3 Subsequent Discussions

The compact brilliance of Poincaré’s discussion [9] was such that it was roughly 50 years before others came to appreciate some of the subtle aspects of his arguments. Meanwhile, a few other discussions emerged of electric charges together with magnetic poles.

#### 3.1 J.J. Thomson

In 1891, Thomson noted [26] that a sheet of electric displacement  $\mathbf{D}$  (parallel to the surface) which moves perpendicular to its surface with velocity  $\mathbf{v}$  must be accompanied by a sheet of magnetic field  $\mathbf{H} = \mathbf{v}/c \times \mathbf{D}$  according to the free-space Maxwell equation  $\nabla \times \mathbf{H} = (1/c) \partial \mathbf{D} / \partial t$ .<sup>10</sup> Then, the motion of the energy density of these sheets implies there is also a momentum density, eqs. (2) and (6) of [26],

$$\mathbf{p}_{\text{EM}}^{(\text{Thomson})} = \frac{\mathbf{D} \times \mathbf{H}}{4\pi c}. \quad (33)$$

In 1893, Thomson transcribed much of his 1891 paper into the beginning of *Recent Researches* [21], adding the remark (p. 9) that the momentum density (33) is closely related to the Poynting vector [25, 32],<sup>11,12</sup>

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}. \quad (34)$$

In vacuum, the field momentum density is simply,

$$\mathbf{p}_{\text{EM}}^{(\text{E-B})} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}, \quad (35)$$

which we consider now. Then, the total field momentum of a system is,

$$\mathbf{P}_{\text{EM}}^{(\text{E-B})} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol}. \quad (36)$$

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<sup>10</sup>Variants of this argument were given by Heaviside in 1891, sec. 45 of [30], and much later in sec. 18-4 of [15], where it is noted that Faraday’s law,  $\nabla \times \mathbf{E} = -(1/c) \partial \mathbf{B} / \partial t$ , combined with the Maxwell equation for  $\mathbf{H}$  implies that  $v = c$  in vacuum, which point seems to have been initially overlooked by Thomson, although noted in sec. 265 of [31].

<sup>11</sup>The idea that an energy flux vector is the product of energy density and energy flow velocity seems to be due to Umov [33], based on Euler’s continuity equation [34] for mass flow,  $\nabla \cdot (\rho \mathbf{v}) = -\partial \rho / \partial t$ .

<sup>12</sup>Thomson argued, in effect, that the field momentum density (4) is related by  $\mathbf{p}_{\text{EM}} = \mathbf{S}/c^2 = u\mathbf{v}/c^2$  [26, 21]. See also eq. (19), p. 79 of [30], and p. 6 of [35]. It turns out that the energy flow velocity defined by  $\mathbf{v} = \mathbf{S}/u$  can exceed  $c$  (see, for example, sec. 2.1.4 of [36] and sec. 4.3 of [37]).

In 1904, Thomson [22, 38, 39, 40] considered the field momentum, and field angular momentum,

$$\mathbf{L}_{\text{EM}}^{(\text{E-B})} = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol}, \quad (37)$$

for various examples, including an electric charge together with a magnetic pole, both at rest. He computed that the field momentum is zero in the latter example according to eq. (35), while the field angular momentum is,

$$\mathbf{L}_{\text{EM}}^{(\text{E-B})} = -\frac{ep}{c} \hat{\mathbf{r}}, \quad (38)$$

according to eq. (36). That is, Thomson clarified in 1904 that the term  $-ep\hat{\mathbf{r}}/c$  in Poincaré's eq. (4) has the physical significance of angular momentum stored in the electromagnetic field of the system. However, Thomson did not reference either Birkeland or Poincaré in his discussion.

Thomson also discussed the case of an electric charge together with either an Ampèrian or Gilbertian magnetic dipole  $\mathbf{m}$ , finding that,

$$\mathbf{P}_{\text{EM}}^{(\text{G})} = 0, \quad \mathbf{P}_{\text{EM}}^{(\text{A})} = \frac{\mathbf{E} \times \mathbf{m}^{(\text{A})}}{c}, \quad (39)$$

Thomson did not compute the electromagnetic field angular momentum for these cases, but the result is,

$$\mathbf{L}_{\text{EM}} = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Vol} = \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{m}}{c}, \quad (40)$$

for both Ampèrian and Gilbertian magnetic dipoles.<sup>13</sup>

Thomson's insights, like those of Poincaré on this topic, were ahead of their time, and also went largely unnoticed for many years.

## 3.2 Størmer

Birkeland's work on the effect of magnetic fields on cathode rays led him to an interest in the effect of terrestrial magnetism on electrons in the Earth's upper atmosphere, in particular on the *aurora borealis* (frequently observable in Norway). Beginning in 1904, a younger colleague, C. Størmer, was inspired by Birkeland to make extensive modeling [41, 42] of the trajectories of electrons in the Earth's magnetic field, approximated as that of a magnetic dipole.

On p. 1 of [41] and p. 210 of [42], Størmer made brief mention of Birkeland's cathode-ray studies, of Poincaré's analysis, and then identified Darboux [10] as the source of this

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<sup>13</sup>The electromagnetic field angular momentum (31) deduced via a Darboux-Poincaré analysis is correct, but the inference (32) about the electromagnetic field momentum for an electric charge together with a Gilbertian magnetic dipole is incorrect. Perhaps Poincaré was wise to avoid any inferences of this type, whose eventual clarification is extremely subtle.

analysis.<sup>14</sup> Størmer was interested in numerical integration of electron trajectories in complex magnetic fields, so did not emphasize the constant vector  $\mathbf{J}$  of Darboux and Poincaré.

The photo below is of Størmer and Birkeland in 1910.



### 3.3 Dirac

Perhaps the most well known paper on magnetic charges is Dirac's 1931 comment [43, 44] that if magnetic poles of strength  $p$  are to be introduced into quantum electrodynamics, while the usual gauge invariance and phase invariance of the wavefunction [45, 46] is to be retained, then,

$$\frac{ep}{c} = \frac{n\hbar}{2} \quad (41)$$

must hold for any physical electric charge  $e$ , where  $n$  is an integer. This gave a perspective on how/why all observed electric charges are integer multiples of the charge of the electron.

Dirac made no mention of Poincaré or Thomson, but noted that Tamm [47] had recently worked out the quantum wavefunction of this system.

### 3.4 Saha

In 1936, Saha published a paper on the possible electromagnetic origin of the mass of the proton and neutron [48]. Section 3, on free magnetic poles, commented on Dirac's paper, that the classical, field angular momentum of an electric charge together with a magnetic pole is  $ep/c$ , so if this is quantized as integer multiples of  $\hbar/2$ , Dirac's relation is obtained. Details of the calculation were not given, and no reference was made to Poincaré or Thomson.

Saha also speculated (p. 146) that the magnetic moment of the neutron might be due to a pair of opposite magnetic poles, apparently unaware that Fermi had argued in 1930 [49]

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<sup>14</sup>Størmer studied under Darboux and Poincaré in 1898-1900.

that this is not the case.<sup>15</sup> An added note discussed the energy levels of  $e^+e^-$ , first computed in 1934 by Mohorovičić [51].

### 3.5 Fierz

In a 1943 commentary on Dirac’s paper, Fierz [52] included a classical analysis of an electric charge together with a magnetic pole. This analysis was presented in vector notation, and identified the constant vector  $\mathbf{J}$  of our eq. (4) as the angular momentum of the system (calling this  $\mathbf{d}$  in eq. (1.2)).

No mention was made of Poincaré or Thomson.

### 3.6 Banderet

In a 1946 commentary on the papers of Dirac and Fierz, Banderet [53] made brief mention of a classical analysis that the half angle of the cone on which the electric charge moves is given by the first version of our eq. (19), which Banderet attributed to Poincaré.<sup>16</sup>

### 3.7 Wilson

In 1949, Wilson published a brief note [54] pointing out that Thomson (1904) had deduced the field angular momentum of an electric charge together with a magnetic pole, commenting that this leads quickly to Dirac’s 1931 quantum condition.

Saha then claimed [55] that he had also pointed this out in 1936 [48].

### 3.8 Ford and Wheeler

In 1959, Ford and Wheeler [56] gave a Hamilton-Jacobi analysis of the scattering of an electric charge by a magnetic pole, and commented on the quantum analyses mentioned above. Poincaré and Thomson were not mentioned, but Fierz was cited.

### 3.9 Lapidus and Pietenpol

The classical scattering of an electric charge by a magnetic pole was given a pedagogic treatment in 1960 by Lapidus and Pietenpol [57], who referred to the quantum treatments mentioned above, but not to the classical analysis of Ford and Wheeler. They made the statement that “the problem has been treated in a somewhat different manner by Poincaré” (who did not explicitly discuss scattering).

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<sup>15</sup>See [50] for a review of the evidence that the permanent magnetism of matter is Ampèrian rather than Gilbertian.

<sup>16</sup>This is the first reference I have found to Poincaré’s paper [9] after the year 1896 (except for the papers of Størmer, which were outside the mainstream of the emerging subject of elementary particle physics).

### 3.10 Nadeau

Nadeau (1960) [58] made a comment on the paper of Lapidus and Pietenpol [57] that the classical motion of an electric charge in the field of a magnetic pole can readily be deduced via vector analysis, without mention of Poincaré or Fierz. He gave arguments equivalent to eqs. (1)-(12) above, which constituted the most complete vectorial discussion of Poincaré's analysis at the time.

### 3.11 Goldhaber

In the Introduction to a 1965 paper on quantum monopoles, Goldhaber [59] mentioned the result (38) for the field angular momentum of an electric charge together with a magnetic pole, attributing this classical result to Wilson [54].

### 3.12 Schwinger

In 1966, Schwinger [60] argued that Dirac's quantum condition should read  $ep/c = n\hbar$  rather than  $n\hbar/2$ . On p. 1089 he said "The discrepancy has arisen from our use of an infinite discontinuity line, in accordance with space-reflection considerations, rather than the semi-infinite line employed by Dirac." He made tangential reference to the semiclassical argument of Wilson.

In a 1968 comment in Schwinger's paper, Peres [61] argued that the constant vector  $\mathbf{J}$  should be written as  $\mathbf{r} \times (\mathbf{p}_{\text{canonical}} - e\mathbf{A}/c) - ep\hat{\mathbf{r}}/c$  in quantum discussions.

### 3.13 Rossi and Olbert

In the 1970 text *Introduction to the Physics of Space*, Rossi and Olbert [62] discussed the motion of an electric charge in the field of a magnetic pole (sec. 2.5) with no references, and without physical interpretation of the constant vector  $\mathbf{J}$  (their eq. (2.80)).

Chapter 3 is an extensive review of the work of Størmer [42].

### 3.14 Kerner

In 1970, Kerner [63] made some remarks about the case of relativistic motion, and referenced Poincaré for the nonrelativistic case (apparently not realizing that Poincaré's analysis holds for any velocity of the charge, so long as the magnetic pole remains fixed, if one interprets the mass  $m$  of the charge as its relativistic mass). He found a second constant vector for the system, and compared this to the Runge-Lenz vector of the Kepler problem; like the Runge-Lenz vector, Kerner's new vector does not appear to have a simple physical interpretation.

Kerner may have been the first, 74 years after Poincaré's paper, to note explicitly that Poincaré's constant vector  $\mathbf{J}$  (our eq. (4)) is the total angular momentum of the system, but he does not claim that Poincaré was aware of this.

### 3.15 Carter and Cohen

In 1973, Carter and Cohen [64] presented a classical discussion, aimed in part in distinguishing the classical and quantum cases. Birkeland, Poincaré and Thomson are referenced. They also did not find a useful physical interpretation for Kerner’s constant vector.

Appendix A of [64] presents a cumbersome calculation of the field angular momentum of a charge plus pole (given much more succinctly by Thomson [39]). Appendix B argues that attributing Dirac’s quantum condition (41) to quantization of the field angular momentum is oversimplified. This theme was continued by Cohen in [65].

### 3.16 Jackson

In sec. 6.13 of the 1975 (2nd) edition (sec. 6.12 of the 3rd (1999) edition) of his text *Classical Electrodynamics*, Jackson [66] gave a derivation, eqs. (6.158)-(6.159), of our eq. (38), referencing J.J. Thomson [40], as part of his discussion of Dirac’s quantum condition.

Jackson began sec. 6.13 with a discussion of an experiment much like that of Birkeland, but with a weak magnetic pole  $p$  and large initial mechanical angular momentum  $mv_0x_0 \gg ep/c$  of an electric charge  $e$  that originated at  $(x_0, 0, z_0)$  for large negative  $z_0$ . Then, the constant vector  $\mathbf{J} = \mathbf{L}_{\text{mech}} + ep\hat{\mathbf{r}}/c = -mv_0x_0\hat{\mathbf{y}} - ep\hat{\mathbf{z}}/c$  is essentially just the initial mechanical angular momentum  $-mv_0x_0\hat{\mathbf{y}}$ , the half angle of Poincaré’s cone is  $\theta \approx 90^\circ$ , and this cone is approximately the  $x$ - $z$  plane. As the electric charge moves on this cone/plane in roughly a straight line, passing the magnetic pole<sup>17</sup> and heading off to large positive  $z$ , the field angular momentum  $ep\hat{\mathbf{r}}/c$  changes from  $-ep\hat{\mathbf{z}}/c$  to  $ep\hat{\mathbf{z}}/c$ . Since  $\mathbf{J} = \mathbf{L}_{\text{mech}} + ep\hat{\mathbf{r}}/c$  is constant, we infer that the change in mechanical angular momentum is  $\Delta\mathbf{L}_{\text{mech}} = 2ep\hat{\mathbf{z}}/c$ . Jackson computed this change, eq. (6.156), by first estimating the change  $\Delta P_y \approx 2ep/cx_0$ , eqs. (6.154)-(6.155), in the mechanical momentum of the charge due to the Lorentz force on it as it passes the magnetic pole.<sup>18</sup>

Thus, Jackson’s example confirms Poincaré’s analysis in the other limit from that considered in sec. 2.2. However, Jackson seemed unaware of Poincaré’s priority here (and Poincaré himself was unaware of such priority).

Jackson found it awkward to associate half-integral quanta of  $\hbar$  with the field angular momentum as an “explanation” of Dirac’s quantum condition (13), and made no reference to Schwinger’s claim [60] that the quantization is actually in integer multiples of  $\hbar$ .

### 3.17 Adawi

In 1975, Adawi [67] gave a short review of computations of the field angular momentum of an electric charge in field of a magnetic pole at rest, mentioning Thomson but not Poincaré.

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<sup>17</sup>The closest approach of the charge to the pole is  $x_0$  in this case, as also found in sec. 2.2 when  $mv_0x_0 \ll ep/c$ . In the language of particle scattering, the cross section for capture of the charge particle by the magnetic pole is zero.

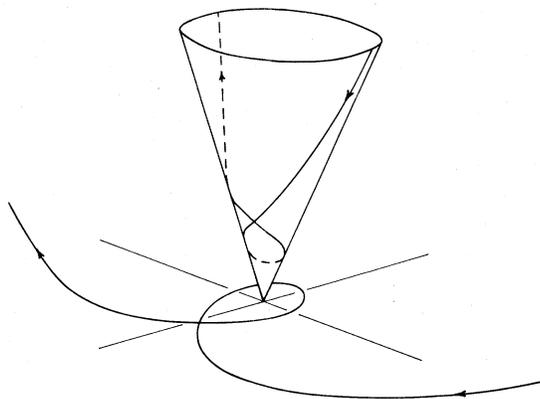
<sup>18</sup>Jackson notes that his calculation (6.155) of the momentum “kick” was given in eq. (2.1) of [59].

### 3.18 Goddard and Olive

In 1978, Goddard and Olive [69] gave a review of magnetic monopoles in gauge field theories, discussing the motion of an electrically charged particle in a radial magnetic field in sec. 2.2, referencing Poincaré.

### 3.19 Boulware *et al.*

In 1976, Boulware *et al.* discussed the quantum scattering of an electric charge by a magnetic pole, including a long sec. 2 on the classical case (for  $mv_0x_0 \ll epc$ ) in which Poincaré, Fierz, Lapidus and Pietenpol, and Nadeau are cited, but not Ford and Wheeler (whose work most closely anticipated theirs). Their fig. 2, shown below, is a nice illustration of the motion of the electric charge on the cone near the point of closest approach to the magnetic pole. However, this figure does not assume that the  $z$ -axis is a generatrix of the cone.



### 3.20 Zia

In 1979, Zia [70] gave a pedagogic discussion of the classical motion of an electric charge in the field of a magnetic pole, citing Dirac, Lapidus and Pietenpol, Nadeau, Ford and Wheeler, Boulware *et al.*, Carter and Cohen, but not Poincaré.

### 3.21 Salam

In his opening address [71] to the 1981 Monopole Conference, Salam stated that *around 1895, Poincaré and J.J. Thomson had published the result that the static field generated by a charge  $+e$  placed at a distance  $r$  from the pole of charge  $+g$  gives rise to an angular momentum  $eg/4\pi$ .* Presumably he was referring to the papers [9, 39] from 1896 and 1904. He also cited Saha [48].

### 3.22 Davis and Perkins

Interest in monopoles received a big boost in the mid 1970's when 'tHooft [72] and Polyakov [73] noted that superheavy monopoles (which obey the Dirac quantum condition (41)) arise in many grand-unified gauge-field theories. Such monopoles are grand-unified partners with

the ordinary fermions, so transitions exist among all of these. Of particular interest is the possibility, noted by Rubakov [74] and Callan [75], that a monopole could interact with a proton, leading to the “decay” of the latter with a cross section typical of the size of the proton, rather than the much smaller cross sections typical of the grand-unified mass/energy scale.<sup>19</sup>

In 1989, Davis and Perkins [76] considered an extension of the analysis of Poincaré (whom they did not mention) to include a charged particle (of mass  $m$  and electric charge  $e$ ) with spin angular momentum  $\mathbf{S}$  and magnetic moment  $\boldsymbol{\mu} = \Gamma e\mathbf{S}/mc$ , where  $\Gamma \approx 1$  for an electron and 2.79 for a proton. Unlike the case of a spinless charged particle which has a nonzero distance of closest approach to the monopole (Poincaré’s 8th equation, our eq. (15)), the claim of [76] is that a charged particle with spin can reach the origin, such that the capture cross section of a pointlike charged particle with spin by a superheavy monopole is nonzero.

To examine this claim, we note that the spatial equation of motion for a charged particle with spin and a superheavy monopole at rest is modified from eq. (2) to become,

$$m\frac{d^2\mathbf{r}}{dt^2} = \frac{e}{c}\frac{d\mathbf{r}}{dt} \times \mathbf{B} + \nabla(\boldsymbol{\mu} \cdot \mathbf{B}) = \frac{ep}{cr^3}\frac{d\mathbf{r}}{dt} \times \mathbf{r} - \frac{\Gamma ep}{mc}\frac{3(\mathbf{S} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{S}}{r^3}. \quad (42)$$

In addition, the magnetic moment obeys the torque equation,

$$\frac{d\mathbf{S}}{dt} = \boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B} = \frac{\Gamma ep}{mcr^2}\mathbf{S} \times \mathbf{r} \equiv \boldsymbol{\Omega}_S \times \mathbf{S}, \quad \text{where} \quad \boldsymbol{\Omega}_S = -\frac{\Gamma ep}{mcr^2}\mathbf{r}. \quad (43)$$

That is, the spin vector  $\mathbf{S}$  precesses about the radial lines of magnetic field from the monopole.

As in sec. 2.1, we take the vector cross product of eq. (42) with  $\mathbf{r}$ ,

$$\begin{aligned} \mathbf{r} \times m\frac{d^2\mathbf{r}}{dt^2} &= \frac{d}{dt} \left( \mathbf{r} \times m\frac{d\mathbf{r}}{dt} \right) = \frac{ep}{cr^3}\mathbf{r} \times \left( \frac{d\mathbf{r}}{dt} \times \mathbf{r} \right) + \frac{\Gamma ep}{mcr^2}\mathbf{r} \times \mathbf{S} \\ &= \frac{ep}{cr}\frac{d\mathbf{r}}{dt} - \frac{ep}{2cr^3}\frac{dr^2}{dt}\mathbf{r} - \frac{d\mathbf{S}}{dt} = \frac{ep}{c}\frac{d}{dt}\frac{\mathbf{r}}{r} + \frac{d\mathbf{S}}{dt}, \\ \mathbf{r} \times m\frac{d\mathbf{r}}{dt} &= \frac{ep}{c}\frac{\mathbf{r}}{r} - \mathbf{S} + \mathbf{J}, \end{aligned} \quad (44)$$

where,

$$\mathbf{J} = \mathbf{r} \times m\frac{d\mathbf{r}}{dt} + \mathbf{S} - \frac{ep}{c}\frac{\mathbf{r}}{r} \quad (45)$$

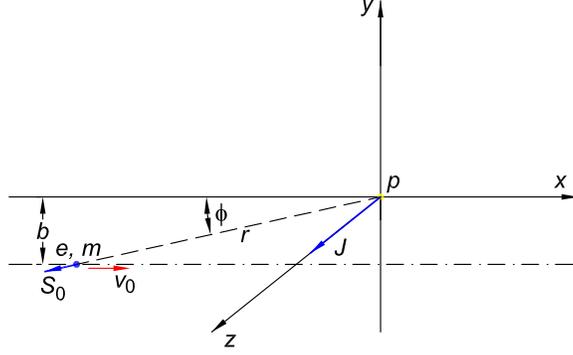
is the constant total angular momentum vector that can be determined from the initial conditions.

*For example, consider a magnetic monopole  $p$  at the origin, and a spin-1/2 particle with electric charge  $e$ , initial velocity  $v_0 \hat{\mathbf{x}}$ , and impact parameter  $b$  in the  $x$ - $y$  plane, as shown in the figure below. Suppose also that the intimal spin direction is away from the monopole,  $\mathbf{S}_0 = \hbar \hat{\mathbf{r}}/2$ , and that the charge and monopole obey the Dirac condition (41),  $ep/c = \hbar/2$ .*

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<sup>19</sup>Even if the monopole has extremely short range interaction with the quarks inside a proton, the extended spatial distribution of these quarks leads to an effective cross section of order  $\pi r_p^2$ , where  $r_p$  is the strong-interaction radius of the proton.

Then, the constant total angular momentum vector is  $\mathbf{J} = mv_0 b \hat{\mathbf{z}}$ , and the initial polar angle of the charge, in spherical coordinates  $(r, \theta, \phi)$  is  $\theta_0 = \pi/2$ .



Poincaré's fourth equation, (5) is now,

$$\hat{\mathbf{r}} \cdot \mathbf{J} = \mathbf{S} \cdot \hat{\mathbf{r}} - \frac{ep}{c}, \quad (46)$$

which is not constant for nonzero spin, and the motion of the charged particle is no longer exactly on the surface of a cone.

In the example above,  $\hat{\mathbf{r}} \cdot \mathbf{J} = (\hbar/2)(\cos \alpha - 1)$  where  $\alpha$  is the angle between  $\mathbf{S}$  and  $\mathbf{r}$ . Initially,  $\hat{\mathbf{r}} \cdot \mathbf{J} = 0$  and the "cone" is simply the  $x$ - $y$  plane. However, as the particle moves angle  $\alpha$  takes on a small nonzero value, the spin vector precesses, and the motion is not precisely in the  $x$ - $y$  plane.

Using eq. (45) in (42),

$$m \frac{d^2 \mathbf{r}}{dt^2} = \frac{ep}{cr^3} \frac{d\mathbf{r}}{dt} \times \mathbf{r} - \frac{\Gamma ep}{mc} \frac{3(\mathbf{J} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{J}}{r^3} - \frac{\Gamma ep}{mcr^3} \mathbf{r} \times m \frac{d\mathbf{r}}{dt} - \frac{2\Gamma e^2 p^2 \mathbf{r}}{mc^2 r^4}, \quad (47)$$

$$\frac{d^2 \mathbf{r}}{dt^2} = \frac{(\Gamma + 1)ep}{mcr^3} \frac{d\mathbf{r}}{dt} \times \mathbf{r} - \frac{\Gamma ep}{m^2 c} \frac{3(\mathbf{J} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{J}}{r^3} - \frac{2\Gamma e^2 p^2 \hat{\mathbf{r}}}{m^2 c^2 r^3}, \quad (48)$$

In a spherical coordinate system  $(r, \theta, \phi)$  with the polar axis along  $\mathbf{J}$  (as for the example above), the component equations of motion for the electric charge  $e$  in the field of the fixed monopole  $p$  are,

$$\ddot{r} - r(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) = -\frac{2\Gamma ep}{m^2 cr^3} J \cos \theta - \frac{2\Gamma e^2 p^2}{m^2 c^2 r^3}, \quad (49)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \cos \theta \sin \theta = \frac{(\Gamma + 1)ep}{mcr^3} r^2 \dot{\phi} \sin \theta - \frac{\Gamma ep}{m^2 cr^3} J \sin \theta, \quad (50)$$

$$r\ddot{\phi} \sin \theta + 2\dot{r}\dot{\phi} \sin \theta + 2r\dot{\theta}\dot{\phi} \cos \theta = -\frac{(\Gamma + 1)ep}{mcr^3} r^2 \dot{\theta}. \quad (51)$$

The last equation can be multiplied by  $r \sin \theta$  to give,

$$r^2 \ddot{\phi} \sin^2 \theta + 2r\dot{r}\dot{\phi} \sin^2 \theta + 2r^2 \dot{\theta} \dot{\phi} \sin \theta \cos \theta = \frac{d}{dt} (r^2 \dot{\phi} \sin^2 \theta) = -\frac{(\Gamma + 1)ep}{mc} \dot{\theta} \sin \theta, \quad (52)$$

which integrates to,

$$r^2 \dot{\phi} \sin^2 \theta = \frac{(\Gamma + 1)ep}{mc} \cos \theta + C, \quad (53)$$

where the constant  $C$  can be evaluated from the initial conditions. For the example above,  $\cos \theta_0 = 0$ ,  $\phi_0 \approx -b/r$ , so  $\dot{\phi}_0 = bv_0/r^2$ , and the initial value of  $r^2\dot{\phi}\sin^2\theta$  is  $bv_0 = C$ .

Equation (50) can be multiplied by  $2r^3\dot{\theta}$  to give,

$$\begin{aligned} 2r^4\ddot{\theta} + 4r^3\dot{r}\dot{\theta}^2 &= \frac{d}{dt} \left( r^4\dot{\theta}^2 \right) = 2r^4\dot{\theta}\dot{\phi}^2 \cos \theta \sin \theta + 2\frac{(\Gamma+1)ep}{mc}r^2\dot{\theta}\dot{\phi}\sin \theta - 2\frac{\Gamma ep}{m^2c}J\dot{\theta}\sin \theta \\ &= 2 \left( \left[ \frac{(\Gamma+1)ep}{mc} \right]^2 + C^2 \right) \frac{\dot{\theta}\cos \theta}{\sin^3 \theta} + 2C\frac{(\Gamma+1)ep}{mc}\dot{\theta} \left( \frac{2}{\sin^3 \theta} - \frac{1}{\sin \theta} \right) - 2\frac{\Gamma ep}{m^2c}J\dot{\theta}\sin \theta, \end{aligned} \quad (54)$$

which integrates to,

$$r^4\dot{\theta}^2 = - \left( \left[ \frac{(\Gamma+1)ep}{mc} \right]^2 + C^2 \right) \frac{1}{\sin^2 \theta} - 2C\frac{(\Gamma+1)ep}{mc}\frac{\cos \theta}{\sin^2 \theta} + 2\frac{\Gamma ep}{m^2c}J\cos \theta + D. \quad (55)$$

Since  $r^4\dot{\theta}^2 \geq 0$ , and in the example above,  $C = bv_0$ ,  $ep/c = \hbar/2$ , and  $\cos \theta_0 = 0$ , the constant of integration obeys  $D \geq (\Gamma+1)^2\hbar^2/4m^2 + b^2v_0^2$ .

Multiplying eq. (49) by  $r^3$  and using eqs. (53) and (55), we now have,

$$\begin{aligned} r^3\ddot{r} &= r^4(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) - \frac{2\Gamma ep}{m^2c}J\cos \theta - \frac{2\Gamma e^2p^2}{m^2c^2} \\ &= D - \left[ \frac{(\Gamma+1)ep}{mc} \right]^2 - \frac{2\Gamma e^2p^2}{m^2c^2} = k = \text{const.} \end{aligned} \quad (56)$$

That is, the radial motion is independent of  $\theta$  and  $\phi$  and the orientation of the magnetic moment  $\boldsymbol{\mu} = \Gamma e\mathbf{S}/mc$  (which this author finds to be a surprising result).

The differential equation  $r^3\ddot{r} = k$  has the general solution (courtesy of Mathematica),

$$r(t) = \frac{\sqrt{k + c_1^2(t + c_2)^2}}{\sqrt{c_1}}, \quad \dot{r} = \frac{dr}{dt} = \frac{c_1(t + c_2)}{r}. \quad (57)$$

The motion has a minimum for  $\dot{r}(-c_2) = 0$ , at which  $r_{\min} = \sqrt{k/c_1}$ . This minimum is physical only if  $k/c_1 \geq 0$ .

We are interested in the case with initial conditions  $r(0) = r_0$  (large) and  $\dot{r}(0) = -v_0$  (negative), for which,

$$c_1 = v_0^2 + \frac{k}{r_0^2}, \quad c_2 = -\frac{r_0v_0}{c_1}, \quad r_{\min} = \sqrt{\frac{kr_0^2}{k + r_0^2v_0^2}} = \sqrt{\frac{k/v_0^2}{1 + k/r_0^2v_0^2}}. \quad (58)$$

The minimum radius  $r_{\min}$  is physical and positive for  $k > 0$  and  $k < r_0^2v_0^2$ .

In the example above, where  $\sin \theta \approx 1$ ,  $ep/c = \hbar/2$  and  $C = bv_0$ , we have that,

$$r^3\ddot{r} = k \geq b^2v_0^2 - \frac{\Gamma\hbar^2}{2m^2}. \quad (59)$$

For a spinless charged particle,  $\Gamma = 0$ , and  $r^3\ddot{r} = k = b^2v_0^2 > 0$ . Here, a minimum radius exists, with value  $r_{\min} = b/\sqrt{1 + b^2/r_0^2} \approx b$ , in agreement with Poincaré's 8th equation (our eq. (15) and footnote 17 (where the impact parameter  $b$  was called  $x_0$ )).

For a particle with charge  $e$ , spin angular momentum  $\hbar/2$ , and a monopole that obeys the Dirac condition  $ep/c = \hbar/2$ , the constant  $k$  can be negative for impact parameter,

$$b < \frac{\sqrt{\Gamma}\hbar}{2mv_0} = \frac{\sqrt{\Gamma}\lambda_m c}{2v_0}, \quad \text{where} \quad \lambda_m = \frac{\hbar}{mc} = \text{Compton wavelength.} \quad (60)$$

In this case, the charged particle reaches the origin<sup>20</sup> and collides with the magnetic monopole there, leading to some interaction (“decay”) of the charged particle in grand-unified theories [74, 75]. The classical capture cross section for this is,

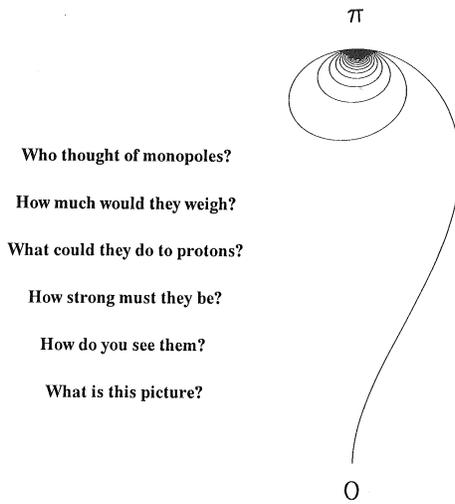
$$\sigma = \pi b^2 = \frac{\pi\Gamma\lambda_m^2 c^2}{4 v_0^2}. \quad (61)$$

For a proton,  $\Gamma = 2.79$  and  $\lambda_p = 1.32 \text{ fm} \approx 1.5r_p$ , so the capture cross section for a relativistic proton with  $v_0 \approx c$  is  $\sigma \approx 1.6\pi r_p^2$ , in agreement with footnote 19.

Of course, the electrons in Birkeland’s experiments [2] had spin, even if this wasn’t yet recognized. However, the low velocities  $v_0$  and large impact parameters ( $b = x_0$ ) in those early experiments were such that the constant  $k$  was positive and there existed a nonzero distance  $r_0$  of closest approach to the magnetic pole.

### 3.23 Goldhaber and Trower

In a 1990 review of literature on magnetic monopoles, Goldhaber and Trower [77] cited Poincaré and Thomson as refs. 98 and 99. Their associated reprint volume [78] has the figure below on its cover.



They describe the figure as a projection of the figure of, for example, sec. 3.19 onto a plane perpendicular to a generatrix of the cone on which the electric charge moves. However, the reverse curve at the bottom of the figure that gives it a form like a ? mark is “artistic license”.

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<sup>20</sup>Formally, if  $\Gamma$  is extremely large such that  $1 + k/r_0^2 v_0^2 < 0$  there again exists a minimum radius of the motion, and the particle would not reach the origin.

### 3.24 Lochak

In 1995, Lochak [79] gave a review in his sec. 3 of the Birkeland-Poincaré discussion, and commented that Thomson was the first to identify the field angular momentum of an electric charge and magnetic pole, although this appeared without interpretation in Poincaré’s 3rd equation (our eq. (4)).

### 3.25 Swarthmore Honors Exam

The motion of an electric charge in the field of a magnetic pole at rest was posed (by D. Griffiths) as prob. 4 on an Honors Exam at Swarthmore College in 1998 [80]. Poincaré was mentioned as having solved this problem.

### 3.26 Sivardière

In 2000, Sivardière [81] presented a pedagogic discussion of the motion of an electric charge in the field of a magnetic pole. In his eq. (6), Sivardière quoted Nadeau’s results (12) [58], without noticing that this describes the angular velocity of the rotating frame which he desires in his sec. 2. Instead, he claimed that his eq. (15) describes the kinetic energy in the rotating frame, failing to see that in this frame there is no mechanical angular momentum, although he comments that the motion is along a straight line through the origin. He then incorrectly concluded (top of p. 186) that  $\Omega = (J - L)/mr^2$ , where  $L$  is the magnitude of the mechanical angular momentum in the lab frame.

Sivardière called the constant vector  $\mathbf{J}$  the Poincaré vector, and attributed the physical interpretation of this as the total angular momentum of the system to Goldhaber [59] and Jackson [66] (although it seems to me that neither mentioned the constant vector  $\mathbf{J}$ ).

### 3.27 Shnir

In a 2005 monograph on magnetic monopoles, Shnir [83] reviewed Poincaré’s argument in sec. 1.1, also mentioning Thomson (but not Darboux), and extended the discussion to the case of dyons in sec. 1.2.

### 3.28 Milton

In 2006, Milton [84] gave a review of searches for magnetic monopoles, with an introductory sec. 2 that mentioned Poincaré and Thomson.

### 3.29 Pilling

In a 2008 e-print, Pilling [85] gave a partial translation, with commentary, of Poincaré’s 1896 paper, claiming that the term  $-ep\hat{\mathbf{r}}/c$  in the constant vector  $\mathbf{J}$  is “spin” angular momentum.

### 3.30 Béch  et al.

In 2014, B ch  et al. [86] reported an experiment somewhat similar to that of Birkeland, claiming the novel discovery that electrons can have orbital angular momentum (aka the “electron vortex state”). No mention was made of Birkeland, Poincar  or Thomson.

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