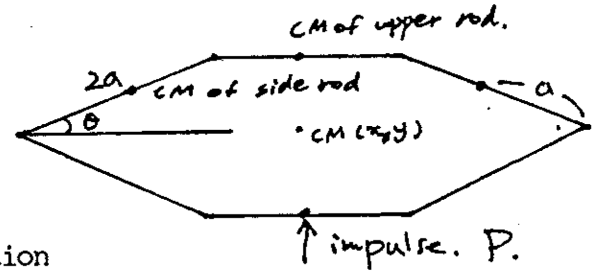
SOLUTION SET FOR PROBLEM SET 4, PH205



1. We will use the coordinate system shown right. The kinetic energy can be decomposed into translational motion of CM,

$$\frac{1}{2}(6m)(\dot{x}^2+\dot{y}^2) \quad m: \text{mass of a rod}$$

and the internal motion about CM. The rotation about CM can also be decomposed into two parts.

For the uppor rod, there is no internal rotation, and the coordinate of its CM is (from CM of hexagon)

$$(0,\ 2a\sin\theta)$$

Thus we can immediately find its kinetic energy.

$$\frac{1}{2}\cdot 4a^2\cos^2\theta\,\dot{\theta}^2$$

Now we consider one of the side rods. The coordinate of its CM can be written as (")

$$(a+a\cos\theta,\ a\sin\theta) \Rightarrow \text{Thus, translational energy is } \frac{1}{2}ma^2\dot{\theta}^2$$

Using the fact that its moment of inertia is $I = 1/3\ ma^2$ and referring to the figure above, we conclude that the rotational energy is,

$$\frac{1}{2}\cdot\frac{1}{3}ma^2\dot{\theta}^2$$

By symmetry, the remaining side rods will have the same kinetic energy. The same relation holds between upper and lower rods. Thus, the total kinetic energy is

$$T = ma^2\left(\frac{8}{3}+4\cos^2\theta\right)\dot{\theta}^2 + \frac{1}{2}\cdot 6m(\dot{x}^2+\dot{y}^2)$$

which is identical to Lagrangian in this case. In our coordinate, the coordinate of the point where impulse is applied can be read from the figure. They read

$$y_P = y - 2a\sin\theta$$

From the Euler-Lagrange equation for impulse type force, we immenately find

$$\Delta\left(\frac{\partial T}{\partial \dot{y}}\right) = 6m\dot{y} = P\cdot\frac{\partial}{\partial y}y_P = P$$

$$\Delta\left(\frac{\partial T}{\partial \dot{\theta}}\right) = ma^2\left(\frac{16}{3}+8\cos^2\theta\right)\dot{\theta} = P\frac{\partial}{\partial\theta}y_P = -2a\cos\theta\cdot P$$

Initially, θ =60. Thus, above equations give, $(\cos^2\theta = \frac{1}{4})$

$$\dot{y} = \frac{P}{6m}, \qquad a\dot{\theta} = -\frac{3}{22}\frac{P}{m}$$

Thus, speed of upper and lower rods v_2 and v_1 is given by

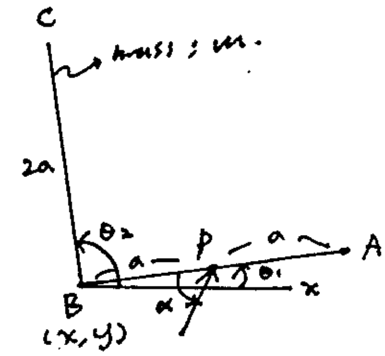
$$v_1 = \dot{y} - 2a\cos\theta\,\dot{\theta} = \frac{P}{m}\left(\frac{1}{6}+\frac{3}{22}\right) = \frac{P}{m}\frac{40}{22\cdot 6}$$

$$v_2 = \dot{y} + 2a\cos\theta\,\dot{\theta} = \frac{P}{m}\left(\frac{1}{6}-\frac{3}{22}\right) = \frac{P}{m}\frac{4}{22\cdot 6}$$

Thus, the ratio is

$$\frac{v_2}{v_1} = \frac{1}{10}$$

2. This problem can easily be solved using elementary method. (In fact elementary method needs less amount of calculations.) However, for the pedagogical purpose I will solve this problem using Lagrangian method.



We will use the coordinate system shown right. The CM coordinate of rod AB is given by

$$(x+a\cos\theta_1,\ y+a\sin\theta_1) \Rightarrow \frac{d}{dt}\vec{x}_{cm}=(\dot{x}-a\sin\theta_1\dot{\theta}_1,\ y+a\cos\theta_1\dot{\theta}_1)$$

which gives the translational energy

$$\frac{1}{2}m(\dot{x}^2+\dot{y}^2+2a\dot{y}\cos\theta_1\dot{\theta}_1-2a\dot{x}\sin\theta_1\dot{\theta}_1+a^2\dot{\theta}_1^2)$$

The rotational energy of CM is given by ($I = 1/3\ ma^2$)

$$\frac{1}{2}\cdot\frac{1}{3}ma^2\cdot\dot{\theta}_1^2=\frac{1}{6}ma^2\dot{\theta}_1^2$$

In our coordinate system rod AB and rod BC is idential except for the subscript. Thus, the Lagrangian can be written as,

$$L=\frac{2}{3}ma^2(\dot{\theta}_1^2+\dot{\theta}_2^2)+m(\dot{x}^2+\dot{y}^2)+ma\dot{y}(\cos\theta_1\dot{\theta}_1+\cos\theta_2\dot{\theta}_2)-ma\dot{x}(\sin\theta_1\dot{\theta}_1+\sin\theta_2\dot{\theta}_2)$$

The coordinates of a point where the impulse is applied can be written as

$$x_p = x+a\cos\theta_1\ ,\ y_p = y+a\sin\theta_1$$

Thus, Euler Lagrange equation for impulse type force gives,

$$\frac{\partial L}{\partial\dot{x}} = m(2\dot{x}-a\sin\theta_1\dot{\theta}_1-a\sin\theta_2\dot{\theta}_2) = P\cos\alpha$$

$$\frac{\partial L}{\partial\dot{y}} = m(2\dot{y}+a\cos\theta_2\dot{\theta}_2+a\cos\theta_2\dot{\theta}_2) = P\sin\alpha$$

$$\frac{\partial L}{\partial\dot{\theta}_1} = \frac{4}{3}ma^2\dot{\theta}_1+ma\dot{y}\cos\theta_1-ma\dot{x}\sin\theta_1 = aP\sin\alpha\cos\theta_1-aP\cos\alpha\sin\theta_1$$

$$\frac{\partial L}{\partial\dot{\theta}_2} = \frac{4}{3}ma^2\dot{\theta}_2+ma\dot{y}\cos\theta_2-ma\dot{x}\sin\theta_2 = 0$$

Using the initial condition $\dot{\theta}_1=\dot{\theta}_2$ (there's no deformation), and θ_1=0 and θ_2=90, we get four equations from the above four equations.

(1) $2\dot{x}-a\dot{\theta}_1 = P/m\ \cos\alpha$

(2) $2\dot{y}+a\dot{\theta}_1 = P/m\ \sin\alpha$

(3) $4/3\ a\dot{\theta}_1+\dot{y} = P/m\ \sin\alpha$

(4) $4/3\ a\dot{\theta}_1-\dot{x} = 0$

(1) + (2) and (3)-(4) give

$$\dot{x}+\dot{y}=\frac{P}{m}\left(\frac{\cos\alpha+\sin\alpha}{2}\right),\ \dot{x}+\dot{y}=\frac{P}{m}\sin\alpha \Rightarrow \cos\alpha=\sin\alpha$$

from which we conclude that α =45. (2)-(3) gives,

$$\dot{y}-\frac{1}{3}a\dot{\theta}_1=0.$$

Thus, from the above equation and (4),

$$\dot{x}=\frac{4}{3}a\dot{\theta}_1\ ,\ \dot{y}=\frac{1}{3}a\dot{\theta}_1$$

The coordiante of A and C can be written as,

$$\vec{A}=(x+2a\cos\theta_1,\ y+2a\sin\theta_1)\qquad \vec{C}=(x+2a\cos\theta_2,\ y+2a\sin\theta_1)$$

which immediately yields their velocity at t=0.

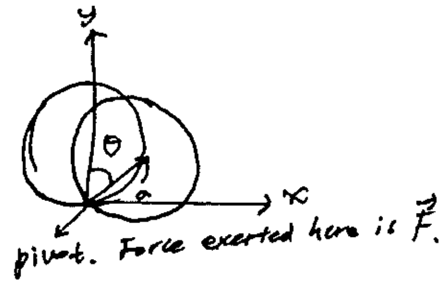
$$\vec{v}_A=(\dot{x},\ \dot{y}+2a\dot{\theta}_1)\ ,\ \vec{v}_c=(\dot{x}-2a\dot{\theta}_2,\ \dot{y})$$

Thus,

$$v_A / v_C = \sqrt{\frac{(\frac{4}{3})^2+(\frac{1}{3}+2)^2}{(\frac{4}{3}-2)^2+(\frac{1}{3})^2}} = \sqrt{\frac{16+49}{4+1}} = \sqrt{\frac{65}{5}} = \sqrt{13}$$

3. The Newton's second law gives, $(r = \text{constant} = a)$

$$\vec F + (-mg\hat y) = m(\ddot r - r\dot\theta^2)\hat r + m(2\dot r\dot\theta + r\ddot\theta)\hat\theta$$



where we used plane polar coordinates. From the torque equation, (about pivot point)

$$a mg \sin\theta = \frac{3}{2} m a^2 \ddot\theta \Rightarrow \ddot\theta = \frac{2}{3}\frac{g}{a}\sin\theta$$

,where we used I = (1 + 1/2) ma^2 (parallel axis theorem), we can also get,

$$\dot\theta^2 = \frac{4}{3}\frac{g}{a}(1-\cos\theta) + \dot\theta^2(0) = \frac{4}{3}\frac{g}{a}(1-\cos\theta)$$

by direct integration. Thus, for arbitrary angle θ , the force can be written as

$$\vec F = mg\hat y - \frac{4}{3}mg(1-\cos\theta)\hat r + \frac{2}{3}mg\sin\theta\,\hat\theta$$

If θ =90, $\hat r = \hat x$ and $\hat\theta = -\hat y$. Thus,

$$\vec F = -\frac{4}{3}mg\hat x + (1-\frac{2}{3})mg\hat y \Rightarrow F = \sqrt{\frac{4^2+1^2}{3^2}}\,mg = \frac{\sqrt{17}}{3}mg$$

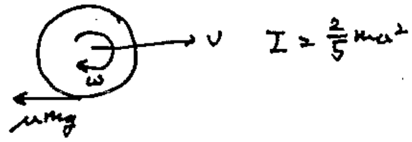
If θ =180, $\hat r = -\hat y$, $\hat\theta = -\hat x$. Consequently,

$$\vec F = (1+\frac{8}{3})mg\hat y \Rightarrow F = \frac{8}{3}mg.$$

4.(a) Follwing the procedure of the previous problem contained in the last problem sets, the rolling without slipping speed of the billiard ball is obtained to be,

$$v = \frac{5v_0 + 2r\omega_0}{7}$$

(For the ball, the force equation for CM is



$$m\ddot x = -\mu mg \Rightarrow v = v_0 - \mu g t$$

and the torque equation about CM is

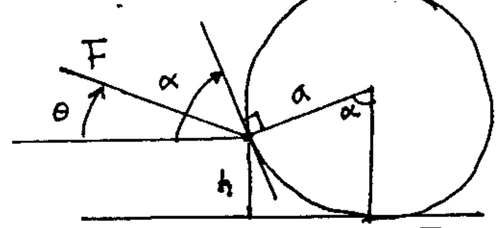
$$I\dot\omega = +\mu mga \Rightarrow \omega = \omega_0 + \frac{\mu mgr}{I}t$$

Requiring rolling without slipping condition yields the time it is achieved. That is,

$$v = a\omega \Rightarrow v_0 - \mu g t = a\omega_0 + \frac{5}{2}\mu g t$$

Putting this into v gives the desired result.)

Consider the figure shown right. Let us assume the shot happens during Δt , with impulse force F.



$$\cos\alpha = \frac{a-h}{a},\quad \sin\alpha = \frac{\sqrt{a^2-(a-h)^2}}{a} = \sqrt{h^2-2ah}/a.$$

Then, the force equation gives

$$F\cos\theta\,\Delta t = m\Delta v$$

whereas the torque equation gives

$$aF\sin(90+\alpha-\theta)\Delta t = \frac{2}{5}ma^2\Delta\omega.$$

We divide two equations to get $(\Delta v = v_0,\ \Delta\omega \simeq \omega_0)$

$$\cos\alpha\frac{\cancel{\cos\theta}}{\cos\theta} + \sin\alpha\frac{\sin\theta}{\cos\theta}) = -\frac{2}{5}\frac{r\omega_0}{v_0}$$

Now we require the returning condition v≤0, which implies $v_0 \le -\frac{2}{5}r\omega_0$. Then, (1) becomes

$$\cos\alpha + \sin\alpha\tan\theta \ge 1$$

$$\tan\theta \ge (1-\cos\alpha)/\sin\alpha = \frac{h}{a}\cdot\frac{a}{\sqrt{h^2-2ah}} = \frac{1}{\sqrt{2a/h-1}}$$

(b) Since there is no friction between two balls, the angular speed can not be changed during the collision process. Furthermore, since the collision is elastic one, the linear momentum and total kinetic energy is conserved. The well-known result of this two conservation law is that, if one ball was initially at last, the other ball stops and the initially stopped ball moves with the same speed. (Since this obviously satisfies energy and momentum conservation, we can verify our assertion.) Thus the initial angular velocity and the speed of ball 1 after impact is,

$$v_0 = 0 \;, \quad \omega_0 = \omega_{1i}$$

Using the formula we derived in (a), the rolling without slipping speed is,

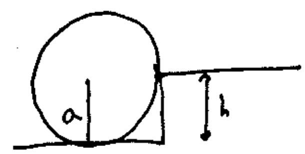
$$v = \frac{5v_0 + 2r\omega_0}{7} = \frac{2}{7} a\,\omega_{1i}$$

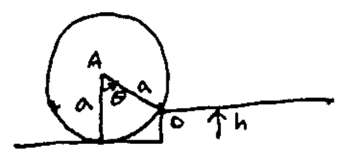
The initial speed for ball 2 is

$$v_0 = v_{1i} \,, \quad \omega_0 = 0$$

which implies the rolling without slipping speed, $v = \frac{5}{7} v_{1i}$.

5. (a) The ball initially moves from left to right and bumps the the bumper. At that time, it gets impulsive force from the bumper and starts to rotate about the point of impact, since there is no slippage at the point. Thus, if the initial angular speed large enough that it can stand vertically, i.e., if the line connecting CM of the ball and the point of impact becomes perpendicular to the plane, it can jump. Using the parallel axis theorem and energy conservation that condition can be expressed as,





(θ is angle btwn OA and $\vec{g}$ direction)

$$\Delta U = mgh = \left(\frac{1}{2}ma^2 + \frac{2}{5}ma^2\right)\omega^2 = \frac{7}{10} m a^2 \omega^2 , \quad \omega = \frac{1}{a}\sqrt{\frac{10}{7}gh}$$

During the impact, the angular momentum about the point of fixture (point of impact) should be continuous since there is no impulse type torque about the point. Thus,

$$m(a-h)v_0 + \frac{2}{5} m a^2 \omega_0 = \frac{7}{5} m a^2 \omega$$

The above condition gives, for $\omega_0 = 0$,

$$v_0 \geq \frac{a^2}{a-h}\cdot\frac{7}{5}\cdot\frac{1}{a}\sqrt{\frac{10}{7}gh} = \frac{a}{a-h}\sqrt{\frac{14gh}{5}} = v_{min}$$

and for $\omega_0 = v_0/a$,

$$\left(\frac{7}{5}a - h\right)v_0 \geq \frac{7}{5} m a^2 \omega \;\Rightarrow\; v_0 \geq \frac{a}{7a-5h}\sqrt{70gh} = v_{min}$$

(b) During the rise of the ball, the energy is conserved. That is

$$E = \frac{1}{2} I \dot{\theta}^2 + mga\cos\theta = mga \;; \text{ constant}, \qquad (1)$$

where ~~at the initial~~ we used the fact that for $v = v_{min}$ case, the ball stops when it reaches $\theta = 0$. (see the above figure for the definition of θ.)

The normal force exerted by the step at the point of contact, N, is obtained from the force equation along the radial direction. Thus, ($\vec{N} = N\hat{r}$, clearly)

$$m\ddot{a} - ma\dot{\theta}^2 = N - mg\cos\theta \;, \quad N = -ma\dot{\theta}^2 + mg\cos\theta$$

Using (1), the normal force can be written as

$$N = mg\left(-\frac{10}{7}(1-\cos\theta) + \cos\theta\right) = \frac{mg}{7}(17\cos\theta - 10)$$

Notice that the ball flies off if the normal force becomes 0 (or negative). As the ball rotates about the point of impact, i.e. as the angle θ decreases from θ_0 (which is defined by $\cos\theta_0 = \frac{a-h}{a}$) to 0, the normal force gets increase. Thus, if the ball is to fly off, it should do that immediately after the time of impact. Consequently, setting $\theta = \theta_0$ in the normal force expression and requiring it to be negative yields the condition

$$17\,\frac{a-h}{a} - 10 \le 0 \quad \Rightarrow \quad h \ge \frac{7}{17}a.$$

for the ball to fly off.

6. In the rest frame of rocket, the energy conservation gives,

$$M_1c^2 = \frac{M_2c^2}{\sqrt{1-(dv/c)^2}} + \frac{dm\,c^2}{\sqrt{1-u^2/c^2}} \qquad (dm\,;\ \text{rest mass of ejected fuel.})$$

Like wise the conservation of momentum gives,

$$0 = \frac{M_2\,dv}{\sqrt{1-(dv/c)^2}} - \frac{dm\cdot u}{\sqrt{1-u^2/c^2}}$$

We neglect the higher order differential terms (like dv^2, ... etc.) and combine two expressions to get

$$dM \equiv M_1 - M_2 = \frac{dm}{\sqrt{1-u^2/c^2}}\,, \quad M_2\,dv = \frac{dm\cdot u}{\sqrt{1-u^2/c^2}} \Rightarrow M_2\,dv = dM\cdot u \Rightarrow M\,dv = u\,dM \qquad (1)$$

The space-time coordinates in two different frame is related by the Lorentz transform

$$\Delta x'_{lab} = \gamma(\Delta x_{rest} + V\Delta t_{rest}) \qquad \gamma = \frac{1}{\sqrt{1-V^2/c^2}}\,.$$

$$\Delta t'_{lab} = \gamma\left(\Delta t_{rest} + \frac{V}{c^2}\Delta x_{rest}\right)$$

By dividing two expressions and recalling the definition of speed, we get

$$\frac{\Delta x'_{lab}}{\Delta t'_{lab}} = \frac{\Delta x_{rest} + V\Delta t_{rest}}{\Delta t_{rest} + \frac{V}{c^2}\Delta x_{rest}} \qquad \Rightarrow \quad v_{lab} = \frac{V + v_{rest}}{1 + \frac{1}{c^2}V\cdot v_{rest}}$$

In the above expression we set $v_{lab} = V + dV$ and $v_{rest} = dv$. Then,

$$V + dV = \frac{V + dv}{1 + \frac{V}{c^2}dv} \approx (V+dv)\left(1 - \frac{V}{c^2}dv\right) \approx V + \left(1 - \frac{V^2}{c^2}\right)dv$$

which can be rearranged to yield

$$\gamma^2\,dV = dv$$

where

$$\gamma^2 = \frac{1}{1 - V^2/c^2}$$

We rewrite the above expression using the exact differential. Thus,

$$c\cdot d\left(\tanh^{-1}\frac{V}{c}\right) = dv \qquad (2)$$

From (1) & (2),

$$d\left(\tanh^{-1}\frac{V}{c}\right) = \frac{u}{c}\frac{dM}{M}$$

Before we integrate above equation, a caution should be mentioned. Clearly

$$dV = V_{final} - V_{initial}$$

However, if you contemplate the definition of dM, you'll find that

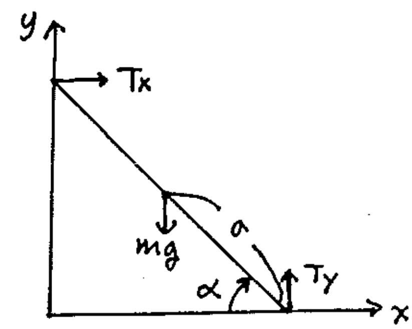
$$dM = M_{initial} - M_{final}$$

Thus, integration of the above equation becomes,

$$\tanh^{-1} V/c - \tanh^{-1} 0 = u/c \ (\ln M_0 - \ln M)$$

since at the initial time, V=0 and $M=M_0$. Taking tanh of the above equation we get,

$$V/c = \tanh \ln (M_0/M)^{u/c}$$
$$= ((M_0/M)^{2u/c} - 1) / ((M_0/M)^{2u/c} + 1)$$



7. Consider the figure in the right side. (Notice that the fact that there is no friction determined the direction of normal forces.) From the force equation, we get

$$T_x = m\ddot{x} , \qquad T_y - mg = m\ddot{y}$$

The torque equation regarding CM as a reference point gives,

$$1/3 \ ma^2 \ddot{\alpha} = a T_x \sin\alpha - a T_y \cos\alpha \qquad (1)$$

where we used the moment of inertia of the rod $I=1/3 \ ma^2$. Since $x=a\cos\alpha$ and $y = a \sin\alpha$ we find

$$T_x = ma \ (-\cos\alpha \cdot \dot{\alpha}^2 - \sin\alpha \ \ddot{\alpha})$$

$$T_y = mg + ma \ (- \sin\alpha \ \dot{\alpha}^2 + \cos\alpha \ \ddot{\alpha})$$

Putting these equations into (1), we get,

$$1/3 \ ma^2 \ddot{\alpha} = - ma^2 \ddot{\alpha} - amg \cos\alpha \quad \text{--->} \quad 4/3 \ a \ddot{\alpha} = - g \cos\alpha$$

Integrating above equation, we get,

$$a \dot{\alpha}^2 = 3/2 \cdot g \ (\sin\alpha_0 - \sin\alpha)$$

where we used the fact that the rod was initially at rest. Using above equations, T_x can be written as,

$$T_x = mg \ (- 3/2 \cos\alpha (\sin\alpha_0 - \sin\alpha) + 3/4 \sin\alpha\cos\alpha)$$
$$= mg \cos \ (- 3/2 \sin\alpha_0 + 9/4 \sin\alpha)$$

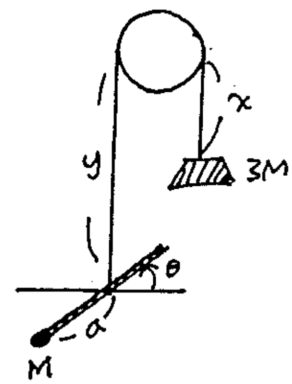
When the rod loses contact, $T_x = 0$. Thus we get

$$\sin\alpha = 2/3 \ \sin\alpha_0$$

8. I will solve this problem using the method of Lagrange multiplier.

We use the coordinate system shown right. In this case, we are given the constraint

$x + y = l$; constant



Now, the coordinate of the mass at the end of the rod is

$(y + a \sin\theta \ , \ a \cos\theta)$

Thus the Lagrangian for that mass can be written as

$$L_M = 1/2\ m\ (\dot{y}^2 + 2\ a \cos\theta\ \dot{\theta}\ \dot{y} + a^2\dot{\theta}^2) + m\ g\ (y + a \sin\theta)$$

Thus, it is trivial to write the total Lagrangian as,

$$L = 1/2\ (3m)\ \dot{x}^2 + 3mg\ x + 1/2\ m\ y^2 + 1/2\ I\dot{\theta}^2 + 1/2\ m\ (\dot{y}^2 + 2a \cos\theta\ \dot{\theta}\dot{y} + a^2\dot{\theta}^2)$$
$$2mg\ y + mg\ a \cos\theta$$

Thus, Euler-Lagrange equation gives,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial}{\partial\theta} L = \frac{d}{dt}(ma\cos\theta\,\dot{y} + ma^2\dot{\theta} + I\dot{\theta}) - mga\cos\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot y}\right) - \frac{\partial}{\partial y} L = 2m\ddot{y} + ma\frac{d}{dt}(\cos\theta\,\dot{\theta}) - 2mg = \lambda \sim \text{Lagrange multiplier}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) - \frac{\partial L}{\partial x} = 3m\ddot{x} - 3mg = \lambda$$

From the constraint we find $\ddot{x} = -\ddot{y}$. And initially, $\theta = 0 = \dot{\theta} = \dot{x} = \dot{y}$ with $I = 1/3\ ma^2$.

Thus above equations can be rewritten as,

$$3m\ddot{x} - 3mg = \lambda \qquad (1)$$
$$-2m\ddot{x} + ma\ddot{\theta} - 2mg = \lambda \qquad (2)$$
$$-m\ddot{x} + \frac{4}{3}ma\ddot{\theta} = mg \qquad (3)$$

(1)-(2) and (3) are

$$5\ddot{x} - g = a\ddot{\theta}$$
$$\ddot{x} + g = \frac{4}{3}a\ddot{\theta}$$

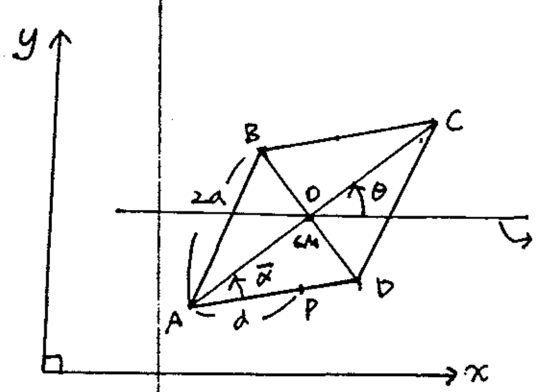
which can be solved to give,

$$a\ddot{\theta} = \frac{18}{17}g, \quad \ddot{x} = \frac{7}{17}g$$

In this case, λ is clearly negative value of tension. (Why?) Thus,

$$T = -\lambda = -3m\ddot{x} + 3\ mg = \left(3 - 3\cdot\frac{7}{17}\right)mg = \frac{30}{17}mg$$

9. We will choose the coordinate shown in the figure.



coordinate of CM : (x, y)

line parallel to x-axis.

The angle θ is the angle between the x-axis and the line connecting two ends of rhombus. (Long diagonal line) The angle $\bar{\alpha}$ is the angle between the diagonal line and the side line. It can be easily shown that these coordinates correspond to translation (x and y), deformation ($\bar{\alpha}$) and rotation (θ). Furthermore, it can be verified that each coordinates are independent in the sense that is explained in the problem. (cf. The explicit form of Lagrangian is

$$L = \frac{4m}{2}(\dot{x}^2+\dot{y}^2) + \frac{8}{3}ma^2\dot{\theta}^2 + \frac{8}{3}ma^2\dot{\bar{\alpha}}^2 \quad \text{where } m \text{ is the mass of a rod.}$$

Try to verify this fact!) Initially $\theta = \bar{\alpha}$. And the coordinate of the point where the impulse is applied is given by

$$y_p = y - (2a-d)\cos\bar{\alpha}\sin\theta - d\sin\bar{\alpha}\cos\theta$$

The impulse type Euler-Lagrange equation can now be written as, (P : impulse, and $\theta = \bar{\alpha}$)

$$\frac{\partial}{\partial\dot{\theta}}L = P(-(2a-d)\cos\bar{\alpha}\cos\bar{\alpha} + d\sin\bar{\alpha}\sin\bar{\alpha})$$

$$\frac{\partial}{\partial\dot{\bar{\alpha}}}L = P(+(2a-d)\sin\bar{\alpha}\sin\bar{\alpha} - d\cos\bar{\alpha}\cos\bar{\alpha})$$

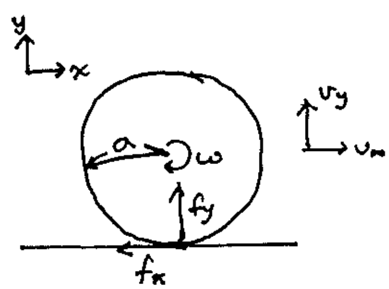
Since the coordinates are not mixed, i.e., they are independent, $\frac{\partial}{\partial\dot{\theta}}L$ is proportional to initial value of rotation rate. Likewise, $\frac{\partial}{\partial\dot{\bar{\alpha}}}L$ is proportional to deformation rate. Thus the no-deformation condition reduces to

$$\frac{\partial}{\partial\dot{\bar{\alpha}}}L = 0, \quad 2a\sin^2\bar{\alpha} = d \Rightarrow d = a(1-\cos 2\bar{\alpha}), \quad d = a(1-\cos\alpha)$$

where we used $2\bar{\alpha} = \alpha$. Likewise, no rotation condition reduces to

$$\frac{\partial}{\partial\dot{\theta}}L = 0, \quad 2a\cos^2\bar{\alpha} = d \Rightarrow d = a(1+\cos 2\bar{\alpha}), \quad d = a(1+\cos\alpha)$$

10. During the time the ball is hitting the rough surface, the equation of motion is,



$$m\,\Delta v_x = f_x\,\Delta t \qquad (1)$$

$$m\,\Delta v_y = f_y\,\Delta t \qquad (2)$$

$$I\,\Delta\omega = -a f_x\,\Delta t \qquad (3)$$

where f_x is frictional force and f_y is normal force. Notice that the motion along y direction is clearly decoupled from x direction and is identical to the usual case of elastic bouncing from the wall. Thus, we can say that $v'_y = -v_y$ (' represents the "after collision") (The friction involved in this case is a static friction since there is no slipping at the point of contact. Thus we should use $f_x < \mu f_y$ rather than $f_x = \mu f_y$. We assume that the surface is sufficiently rough so that μ is very large.) Furthermore, we have the energy conservation since the collision is an elastic one.

$$\frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv_x'^2 + \frac{1}{2}mv_y'^2 + \frac{1}{2}I\omega'^2 \qquad (4)$$

From (1) and (3), we also have

$$m(v_x' - v_x) = -\frac{I}{a}(\omega' - \omega) \qquad (5)$$

(cf. $v_y' = v_y$)

We rewrite (4) and (5) as follows. (I = 2/5 ma^2 and k denotes 2/5)

$$\begin{cases} v_x' - v_x = -k\,(a\omega' - a\omega) \\ \frac{1}{2} m (v_x'^2 - v_x^2) = -\frac{1}{2} mk (a^2\omega'^2 - a^2\omega^2) \Rightarrow \frac{1}{2}(v_x' - v_x)(v_x' + v_x) = \frac{-k}{2}(a\omega' - a\omega)(a\omega' + a\omega) \end{cases}$$

$$\Rightarrow \begin{cases} v_x' - v_x = -k(a\omega' - a\omega) \\ v_x' + v_x = a(\omega' + \omega) \end{cases} \quad \Rightarrow \begin{cases} v_x' - a\omega' = a\omega - v_x \\ v_x' + ka\omega' = ka\omega + v_x \end{cases}$$

$$\therefore \begin{pmatrix} a\omega' \\ v_x' \end{pmatrix} = \begin{pmatrix} \frac{k-1}{k+1} & \frac{2}{k+1} \\ \frac{2k}{k+1} & -\frac{k-1}{k+1} \end{pmatrix} \begin{pmatrix} a\omega \\ v_x \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} & \frac{10}{7} \\ \frac{4}{7} & +\frac{3}{7} \end{pmatrix} \begin{pmatrix} a\omega \\ v_x \end{pmatrix}$$

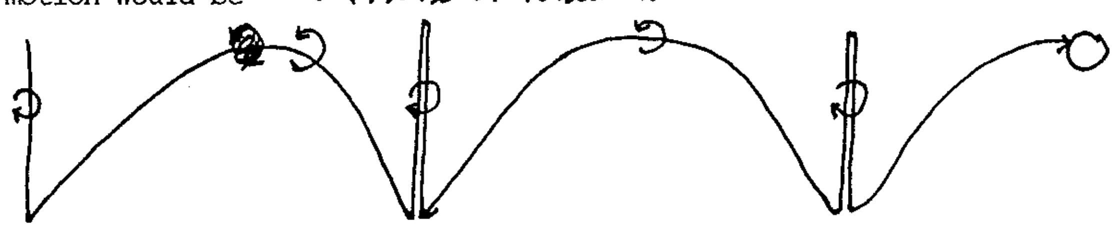
Thus, if, initially, $a\omega = v_0$ & $v_x = 0$ then, after one bounce

$$\begin{pmatrix} a\omega' \\ v_x' \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} & \frac{10}{7} \\ \frac{4}{7} & \frac{3}{7} \end{pmatrix} \begin{pmatrix} v_0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} v_0 \\ \frac{4}{7} v_0 \end{pmatrix}$$

After two bounces, it becomes,

$$\begin{pmatrix} a\omega'' \\ v_x'' \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} & \frac{10}{7} \\ \frac{4}{7} & \frac{3}{7} \end{pmatrix} \begin{pmatrix} -\frac{3}{7} v_0 \\ \frac{4}{7} v_0 \end{pmatrix} = \begin{pmatrix} v_0 \\ 0 \end{pmatrix}$$

After the second bounce, the initial status is reproduced. Thus the graph of the whole motion would be (↻: sense of rotation.)



For the back-and-forth motion to be possible, the velocity and the sense of rotation should be reversed after a bounce. Thus,

$$\begin{pmatrix} a\omega' \\ v_x' \end{pmatrix} = -\begin{pmatrix} a\omega \\ v_x \end{pmatrix}$$

which reduces into,

$$-\begin{pmatrix} a\omega \\ v_x \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} & \frac{10}{7} \\ \frac{4}{7} & \frac{3}{7} \end{pmatrix} \begin{pmatrix} a\omega \\ v_x \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{4}{7} & \frac{10}{7} \\ \frac{4}{7} & \frac{10}{7} \end{pmatrix} \begin{pmatrix} a\omega \\ v_x \end{pmatrix} = 0$$

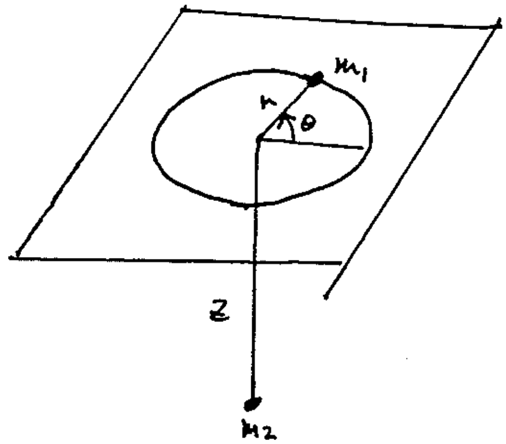
Thus,

$$4a\omega + 10 v_x = 0 \Rightarrow v_x = -\frac{2}{5} a\omega.$$

(cf. Problem 25. looks like a typical eigenvalue problem of the matrix. What is the other eigenvalue and what corresponds to eigenvector motion?)

11. Let us try to solve this problem using Lagrangian.

(a) Referring to the right figure, the Lagrangian can be written as,



$$L = \tfrac{1}{2}m_1(\dot r^2 + r^2\dot\theta^2) + \tfrac{1}{2}m_2\dot z^2 + m_2 g z$$
$$= \tfrac{1}{2}(m_1+m_2)\dot r^2 + \tfrac{1}{2}m_1 r^2\dot\theta^2 - m_2 g r + m_2 g l$$

where we used constraint $r + z = l$; constant. Now, the Euler-Lagrange equation gives,

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot\theta}L\right) - \frac{\partial}{\partial\theta}L = \frac{d}{dt}(m_1 r^2\dot\theta) = \frac{d}{dt}L = 0$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot r}L\right) - \frac{\partial}{\partial r}L = (m_1+m_2)\ddot r - m_1 r\dot\theta^2 + m_2 g = 0$$

The first equation asserts that the initial angular momentum L_0 is conserved. The second equation can be written as,

$$(m_1+m_2)\ddot r = \frac{L_0^2}{m_1 r^3} - m_2 g$$

which is the equation of motion. From the fact $F = -\ V_{eff}$, we get V_{eff} by the direct integration.

$$V_{eff} = -\int dr\left(\frac{L_0^2}{m_1 r^3} - m_2 g\right) = m_2 g r + \frac{L_0^2}{2m_1 r^2}$$

The equillibrium point is determined by,

$$\left.\frac{\partial}{\partial r}V_{eff}\right|_{r=r_0} = 0 \quad\Rightarrow\quad r_0^{\,3} = \frac{L_0^2}{m_1 m_2 g}$$

Thus the effective spring constant is given by, $\left(\because V_{eff} = V_{eff}(r_0) + \overset{0}{\overbrace{\frac{\partial}{\partial r}V_{eff}}}(r-r_0) + \frac{1}{2}\frac{\partial^2}{\partial r^2}V_{eff}\,(r-r_0)^2 + \cdots\right)$

$$k_{eff}^2 = \frac{\partial^2}{\partial r^2}V_{eff} = 3\frac{L_0^2}{m_1 r_0^4} = 3\frac{m_2 g}{r_0} \quad\Rightarrow\quad k_{eff} = \sqrt{\frac{3m_2 g}{r_0}}$$

From the form of the Lagrangian above, we see that the effective mass is $m_1 + m_2$. Thus, the frequency for small radial oscillation is

$$\omega = \sqrt{\frac{3m_2 g}{(m_1+m_2)r_0}}$$

(b) In terms of the relation derived above, the total initial energy for circular motion is given by

$$E = \frac{L_0^2}{2m_1 r_0^2} + m_2 g r_0 - m_2 g l = \frac{3}{2}m_2 g r_0 - m_2 g l$$

The second term is a constant during the whole motion. The motion comes to rest when the first part vanishes. The entire motion would look like a spiral one and the radius of the circular motion decreases due to the damping effect by the dust. From

$$dW = F\cdot ds$$

we find the total distance traveled by the mass is,

$$S_{tot} = \frac{\Delta E}{F_{fric}} = \frac{3}{2}\frac{m_2 g r_0}{\mu m_1 g} = \frac{3}{2\mu}\frac{m_2}{m_1}r_0$$

since the frictional force is a constant. Assuming the orbit nearly remains circular at all times, we find the expression for energy hols always, approximately. Thus, from the relation,

$$\frac{d}{dt}E = +\vec F_{fric}\cdot\vec v \approx -\mu m_1 g\, r_0\dot\theta = -\mu m_1 g\cdot\sqrt{\frac{m_2}{m_1}g}\; r_0^{1/2}$$

$$(\vec v \approx r_0(t)\dot\theta\,\hat\theta)$$

we get,

$$\frac{3}{2} m_2 g \frac{d}{dt} r_0 = -\mu m_1 g \sqrt{\frac{m_2}{m_1} g}\; r_0^{1/2} \Rightarrow \frac{dr_0}{r_0^{1/2}} = -\frac{2}{3} \mu \sqrt{\frac{m_1}{m_2} g}\, dt$$

We integrate above equation under the initial condition $r_0 = 0$ at $t = 0$. From the notion that the motion stops at $r_0 = 0$, we get the total trip time Δt,

$$2\left(r_0^{1/2}(t) - r_0^{1/2}\right) = -\frac{2}{3} \mu \sqrt{\frac{m_1}{m_2} g}\; t, \quad t_0 = \frac{3}{\mu} \sqrt{\frac{m_2 r_0}{m_1 g}}$$

(cf. By dividing $s_{tot}/\Delta t = \frac{1}{2} \sqrt{\frac{m_2}{m_1} g r_0} = \frac{r_0 \dot{\theta}}{2}$, we see that average moving speed was one half of the initial circulating speed, which makes sense.)

P.S. You might have noticed that I have not given you the solution for optional problems and "chanllenge problem". The opportunity is open to you during the whole semester. Please feel free to submit the solution of the problems anytime during the whole semester. Then, I will be gladly grading them. The answers for those problems will be given to you at the end of the semester!