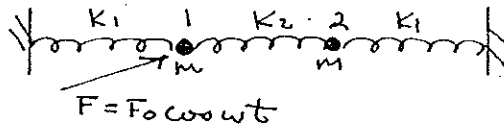


DUE: TUESDAY NOV 20, 1990 ; MAXIMUM RECORDED SCORE = 80 POINTS

① COUPLED, DAMPED, FORCED OSCILLATORS

OUR SYSTEM OF 2 EQUAL MASSES CONNECTED BY 3 SPRINGS



IS SUBJECT TO A DRIVING FORCE ON MASS 1 ONLY, $F = F_0 \cos \omega t$.
 IN ADDITION THERE ARE DAMPING FORCES ACTING ON THE TWO MASSES: $F_1 = -b \dot{x}_1$, AND $F_2 = -b \dot{x}_2$. THE DAMPING FORCES WILL PREVENT THE AMPLITUDES FROM DIVERGING IN CASE ω IS THAT OF A RESONANT FREQUENCY.

COUPLED

DEMONSTRATE THIS BY TRANSFORMING THE EQUATIONS OF MOTION OF x_1 AND x_2 INTO DE-COUPLED EQUATIONS OF MOTION OF THE NORMAL COORDINATES q_1 AND q_2 . TRANSFORM THE SOLUTIONS BACK TO x_1 AND x_2 TO SHOW

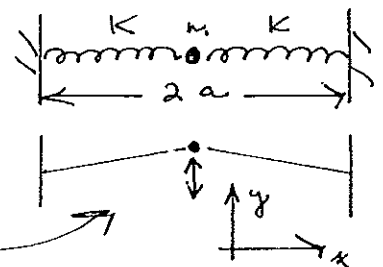
$$x_1 = \frac{F_0}{M} \frac{(\omega_0^2 - \omega^2 + 2i\beta\omega)}{(\omega_1^2 - \omega^2 + 2i\beta\omega)(\omega_2^2 - \omega^2 + 2i\beta\omega)} e^{i\omega t}$$

$$x_2 = \frac{F_0}{2M} \frac{(\omega_2^2 - \omega_1^2)}{\omega_1^2 - \omega^2 + 2i\beta\omega} \frac{e^{i\omega t}}{(\omega_2^2 - \omega^2 + 2i\beta\omega)}$$

WHERE $2\beta = \frac{b}{M}$ $\omega_0^2 = \frac{K_1 + K_2}{M}$ $\omega_1^2 = \frac{K_1}{M}$ $\omega_2^2 = \frac{K_1 + 2K_2}{M}$

HINT: GUESS THE NORMAL COORDS.

② A PARTICLE OF MASS M IS CONNECTED TO 2 FIXED POINTS BY 2 SPRINGS OF CONSTANT K AND REST LENGTH l_0 . THE FIXED POINTS ARE DISTANCE $2a$ APART



CONSIDER TRANSVERSE OSCILLATIONS

a) SHOW THAT IF $a < l_0$ THERE ARE 3 POINTS OF EQUILIBRIUM, AND THAT THE FREQUENCY OF STABLE SMALL OSCILLATIONS

$$\omega = \sqrt{\frac{2K}{M} \left(1 - \frac{a^2}{l_0^2}\right)}$$

b) SHOW THAT IF $a > l_0$ THERE IS ONLY ONE EQUILIBRIUM POINT, AND $\omega = \sqrt{\frac{2K}{M} \left(1 - \frac{l_0}{a}\right)}$ FOR SMALL OSCILLATIONS

(REMEMBER, WE CONSIDER ONLY TRANSVERSE OSCILLATIONS IN THIS PROBLEM.)

② c) IF $a = l_0$ SHOW THAT THE POTENTIAL IS $V(y) = \frac{ky^4}{4l_0^2} + \dots$

USE DIMENSIONAL ANALYSIS TO SHOW THAT PERIOD $\sim \frac{1}{\text{AMPLITUDE}}$.

THEN USE THE METHOD OF SUCCESSIVE APPROXIMATIONS TO SHOW THAT THE PERIOD IS $T \sim 4\pi \frac{\sqrt{3}}{3} \frac{l_0}{A} \sqrt{\frac{m}{k}}$ WHERE $A = \text{AMPLITUDE}$.

THE EXACT ANSWER TURNS OUT TO BE $T = (7.42 \dots) \frac{l_0}{A} \sqrt{\frac{m}{k}}$.

③ A MASS m IS ATTACHED TO A SPRING OF CONSTANT k WHICH IS FIXED AT ITS OTHER END. THE MASS SLIDES (ALONG THE AXIS OF THE SPRING) ON A TABLE, AND IS SUBJECT TO SLIDING FRICTION: $\vec{F} = -\mu mg \hat{v} = -\mu mg \cdot (\text{SIGN OF VELOCITY})$

LET $\omega_0^2 = k/m$. AT $t=0$, $x = A_0$, $\dot{x} = 0$, $A_0 \gg \mu g / \omega_0^2$

a) CONSIDER THE EXACT SOLUTION FOR THE MOTION IN HALF PERIODS TO CALCULATE THE LOSS OF AMPLITUDE. SHOW THAT THE AMPLITUDE WILL DROP TO $\mu g / \omega_0^2 \sim 0$ IN TIME $t \sim \frac{\pi}{2} \frac{A_0 \omega_0}{\mu g}$

b) USE THE METHOD OF AVERAGES TO SHOW THAT THE MOTION IS APPROXIMATELY $x = \left(A_0 - \frac{2\mu g t}{\pi \omega_0} \right) \cos \omega_0 t$

④ MANY DAMPING FORCES BEHAVE LIKE $F = -b\dot{x}^2$ CONSIDER AN OSCILLATOR SUBJECT TO SUCH A FORCE

$$\ddot{x} + \omega_0^2 x = -\beta \dot{x} |\dot{x}|$$

a) USE THE METHOD OF SUCCESSIVE APPROXIMATIONS OVER A HALF PERIOD. SHOW THAT AFTER A HALF PERIOD THE AMPLITUDE HAS BEEN REDUCED FROM A_0 TO $A_0 \left(1 - \frac{4\beta A_0}{3} \right)$

b) USE THE METHOD OF AVERAGES TO SHOW $x \sim a(t) \cos \omega_0 t$ WHERE $\frac{1}{a} = \frac{1}{A_0} + \frac{4\beta \omega_0 t}{3\pi}$ WHICH ALSO LEADS TO THE RESULT OF PART a).

⑤ RECALL PROBS ⑨ & ⑩, SET 5. WE NOW CONSIDER A CENTRAL FORCE $F = -\frac{\alpha}{r^2} - \frac{\beta}{r^4}$

THE ORBIT EQUATION IS THEN $\frac{d^2 u}{d\theta^2} + u = \frac{M\alpha}{L^2} + \frac{\beta M u^2}{L^2}$ ($u = \frac{1}{r}$)

IF $\beta = 0$ THE SOLUTION IS OUR STANDARD ELLIPSE $u = \frac{M\alpha}{L^2} (1 + e \cos \theta)$

FOR $\beta \neq 0$ WE WISH TO DEVELOP A METHOD OF AVERAGES BY SUPPOSING

$$u = \frac{M\alpha}{L^2} (1 + \epsilon(\theta) \cos \phi(\theta))$$

AND ALSO $\frac{du}{d\theta} = -\frac{M\alpha}{L^2} \epsilon \sin \phi$

SUPPOSING $d\epsilon/d\theta$ IS SMALL, SHOW THAT

$$d\phi/d\theta = 1 - \frac{L^2}{2\pi M\alpha \epsilon} \int_0^{2\pi} f \cos \phi d\phi$$

$$d\epsilon/d\theta = -\frac{L^2}{2\pi M\alpha} \int_0^{2\pi} f \sin \phi d\phi$$

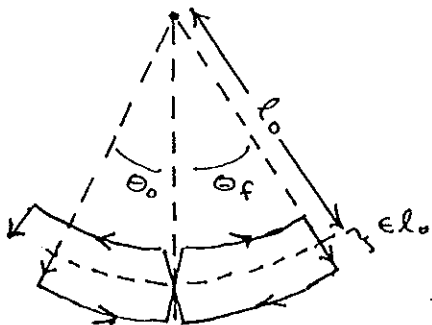
THIS METHOD ALLOWS US TO CONSIDER SMALL OSCILLATIONS ABOUT THE ELLIPSE, RATHER THAN ABOUT A CIRCULAR ORBIT.

SHOW THAT FOR f AS GIVEN, $u \approx \frac{M\alpha}{L^2} (1 + \epsilon \cos (1 - \frac{\alpha\beta M^2}{L^4}) \theta)$

THIS IS A SLOWLY PRECESSING ELLIPSE! THE ANGULAR VELOCITY OF THE PRECESSION OF THE ELLIPSE IS $\omega = \frac{\alpha\beta M^2}{L^4} \Omega$, WHERE $\Omega = \frac{2\pi}{T}$

IS THE AVERAGE ANGULAR VELOCITY OF THE ORBITAL MOTION.

⑥ THE SWING



IF ONE 'PUMPS' A SWING THE AMPLITUDE OF THE OSCILLATION CAN BE INCREASED.

a) AS A SIMPLE MODEL OF THE PUMPING ACTION CONSIDER A SIMPLE PENDULUM WHOSE C.M. IS LOWERED BY $2\epsilon l_0$ WHEN THE AMPLITUDE IS MAXIMUM, AND RAISED BY $2\epsilon l_0$ WHEN $\theta = 0$.

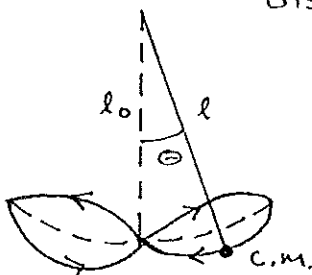
USE ELEMENTARY METHODS TO SHOW THAT DURING ONE HALF PERIOD, $\theta_f^2 = \theta_0^2 \left(\frac{1+\epsilon}{1-\epsilon}\right)^3$

AND THAT IF ϵ IS SMALL THIS LEADS TO $\theta_{MAX} \approx \theta_0 e^{\frac{3\epsilon \omega_0 t}{\pi}}$ ($\omega_0 = \sqrt{\frac{g}{l_0}}$)

HINT: WHAT IS CONSERVED?

⑥ b) IN ANOTHER MODEL OF PUMPING, SUPPOSE THE DISTANCE FROM PIVOT TO C.M. VARIES LIKE

$$l = l_0 (1 + \epsilon \sin 2\omega_0 t)$$

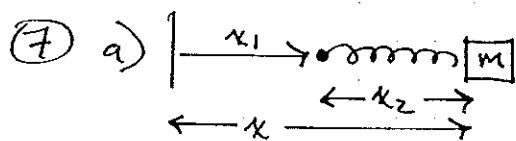


DERIVE THE EQUATION OF MOTION IN θ .

FOR SMALL ϵ , WE SUPPOSE THE SOLUTION WILL BE LIKE $\theta(t) \sim a(t) \cos \omega_0 t$

WHERE $a(t)$ IS SLOWLY VARYING.

USE A METHOD OF SUCCESSIVE APPROXIMATIONS, OR A METHOD OF AVERAGES TO SHOW $\theta \sim \theta_0 e^{\frac{3\epsilon \omega_0 t}{4}} \cos \omega_0 t$ IN THIS CASE.



MASS M IS CONNECTED TO A SPRING OF CONSTANT K , REST LENGTH l_0 .

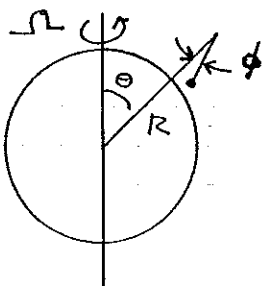
THE OTHER END OF THE SPRING IS

FORCED TO MOVE ACCORDING TO $x_1 = a \cos \omega t$. GO TO THE ACCELERATED FRAME WITH ORIGIN AT x_1 TO SOLVE FOR $x_2(t)$.

THEN SHOW $x = l_0 + \frac{a \omega_0^2}{\omega_0^2 - \omega^2} \cos \omega t$. (+ $A \cos(\frac{K}{M} t + \phi)$ IF NO DAMPING)

b) A PARTICLE HAS VELOCITY v ON A SMOOTH HORIZONTAL PLANE. SHOW THAT THE PARTICLE WILL MOVE IN A CIRCLE, DUE TO THE ROTATION OF THE EARTH, AND FIND THE RADIUS OF THE CIRCLE. (YOU MAY IGNORE TERMS IN Ω_e^2 .)

⑧ a)



A PLUMB LINE DOES NOT POINT DIRECTLY TOWARDS THE CENTER OF THE EARTH DUE TO CENTRIFUGAL FORCE. SHOW THAT THE ANGLE OF DEFLECTION IS

$$\tan \phi = \frac{\sin \theta \omega \Omega}{\frac{g}{\Omega^2 R} - \sin^2 \theta} \quad \left(\frac{g}{\Omega^2 R} \sim 290 \right)$$

b) CENTRIFUGAL BULGE WE MIGHT EXPECT THAT THE SURFACE OF THE EARTH IS ALWAYS PERPENDICULAR TO THE PLUMB LINE (AT LEAST THE WATER SURFACE). THIS REQUIRES A BULGE AT THE EQUATOR. THE SURFACE WILL BE \perp TO THE PLUMB LINE IF IT IS AN EQUIPOTENTIAL OF THE EFFECTIVE POTENTIAL OF GRAVITY + CENTRIFUGAL FORCE.

SHOW THAT THIS LEADS TO THE EQUATION FOR THE SURFACE

$$R^3 \sin^2 \theta = \frac{2GM}{\Omega^2} \left(\frac{R}{R_p} - 1 \right)$$

WHERE R_p = RADIUS AT THE POLE.

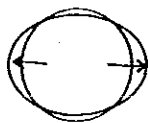
WRITE $R_E/R_P = 1 + \epsilon$ WHERE R_E = RADIUS AT EQUATOR

TO SHOW $\epsilon \sim \frac{1}{580}$

EXPERIMENTALLY, $\epsilon \sim \frac{1}{297}$

(ASSUME THE GRAVITATIONAL POTENTIAL OF THE BULGING EARTH IS THE SAME AS THAT OF A SPHERICAL EARTH.)

⑨ TIDAL BULGE THE SHAPE OF THE EARTH IS ALSO DISTORTED BY THE PULL OF THE MOON AND THE SUN. ROUGHLY, THE MOON PULLS ON THE NEAR SIDE OF THE EARTH MORE THAN ON THE CENTER AND STILL MORE THAN ON THE FAR SIDE. THIS LEADS TO A BULGE FACING THE MOON, AS WELL AS ONE ON THE SIDE OPPOSITE THE MOON.

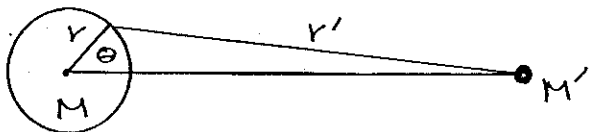


MOON



IGNORE THE EFFECT OF CENTRIFUGAL FORCE DUE TO THE EARTH'S ROTATION IN THIS PROBLEM.

CONSIDER A FRAME WITH THE CENTER OF THE EARTH AT REST. CONSTRUCT THE POTENTIAL DUE TO THE COMBINED GRAVITATIONAL EFFECT OF THE EARTH AND THE MOON ON A PARTICLE AT THE SURFACE OF THE EARTH. USE A SPHERICAL COORD SYSTEM.



NOTE THAT THIS FRAME IS AN ACCELERATED FRAME, SO THERE IS A FICTITIOUS FORCE NEEDED TO HOLD THE EARTH'S C.M. FIXED. SINCE THE EARTH IS RIGID THIS FICTITIOUS FORCE IS THE SAME ON ALL PARTICLES (OF MASS m) THROUGHOUT THE EARTH. BE SURE TO INCLUDE THE POTENTIAL OF THIS FICTITIOUS FORCE.

SHOW
$$\frac{V}{m} \sim -\frac{GM}{r} - \frac{GM'}{R} + \frac{3}{2} \frac{GM' r^2}{R^3} \left(\frac{1}{3} - \cos^2 \theta \right)$$

TO ORDER $(r/R)^2$

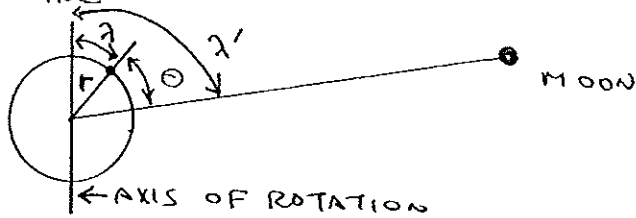
EVALUATE ϵ IN THE EXPRESSION FOR AN EQUIPOTENTIAL

$$r \sim r_0 \left(1 + \epsilon \left(\cos^2 \theta - \frac{1}{3} \right) \right)$$

WHERE r_0 = MEAN RADIUS OF THE EARTH. THIS IMPLIES A BULGE OF 1 FOOT.

BECAUSE THE EARTH IS TURNING ON ITS AXIS, AND THE MOON IS ORBITING AROUND THE EARTH, THE LOCATION OF THE BULGE CHANGES WITH TIME.

CONSIDER A COORDINATE SYSTEM WITH Z AXIS ALONG THE EARTH'S AXIS OF ROTATION, AND X & Y AXES ROTATING WITH THE EARTH. LET (λ, ϕ) BE THE SPHERICAL COORDS OF AN ARBITRARY POINT ON THE EARTH'S SURFACE, AND (λ', ϕ') BE THE COORDS OF THE MOON



SHOW THAT
$$\frac{Y}{Y_0} = 1 + \epsilon \left\{ \begin{aligned} &\frac{3}{2} (\cos^2 \lambda - \frac{1}{3}) (\cos^2 \lambda' - \frac{1}{3}) \\ &+ \frac{1}{2} \sin^2 \lambda \sin^2 \lambda' \cos 2(\phi - \phi') \\ &+ \frac{1}{2} \sin(2\lambda) \sin(2\lambda') \cos(\phi - \phi') \end{aligned} \right\}$$

THUS WHILE THE TOTAL EFFECT IS $Y = Y_0 (1 + \epsilon (\cos^2 \theta - \frac{1}{3}))$, AN OBSERVER AT A FIXED POINT (λ, ϕ) ON THE EARTH CAN DO A 'FOURIER ANALYSIS' OF THE BULGE INTO 3 TIME-VARYING TIDES WITH 3 DIFFERENT PERIODS.

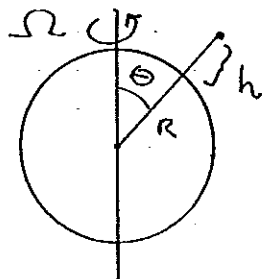
WHAT IS THE PERIOD OF EACH OF THESE TIDES? WHICH ONE CORRESPONDS TO THE COMMON CONCEPTION OF 'THE TIDE'?

IN THE FOLLOWING PROBLEMS CONSIDER THE EARTH AS A SPHERE.

- ⑩ a) A PARTICLE FALLS FROM HEIGHT h ABOVE THE EARTH'S SURFACE ($h \ll R$) AT POLAR ANGLE θ . TO 1ST ORDER IN $\Omega = \text{EARTH'S ROTATION FREQUENCY}$, SHOW THAT THE DEFLECTION IS

$$d = \frac{2}{3} \sqrt{\frac{2h^3}{g}} \Omega \sin \theta$$

IN WHAT DIRECTION? ($g = \text{CONSTANT}$)



- b) SUPPOSE YOU JUMP UP TO HEIGHT h AND THEN FALL BACK. SHOW THAT THE CORIOLIS FORCE CAUSES YOU TO LAND A DISTANCE $d = \frac{8}{3} \sqrt{\frac{2h^3}{g}} \Omega \sin \theta$ FROM WHERE YOU STARTED.

IN WHAT DIRECTION IS d ? WOULD THIS BE DIFFERENT IN THE SOUTHERN HEMISPHERE?

(10) c) USE CONSERVATION OF ANGULAR MOMENTUM AND A NON-ACCELERATED POINT OF VIEW TO CALCULATE THE CHANGE IN AZIMUTHAL ANGLE ϕ DURING YOUR JUMP. SHOW THAT THIS LEADS TO THE SAME DISPLACEMENT AS IN PART b) ($h \ll R$)

(11) A GUN IS LOCATED ON THE EARTH AT POLAR ANGLE θ . IN THE ABSENCE OF THE CORIOLIS FORCE, A SHOT WOULD LAND DISTANCE D AWAY, HAVING RISEN TO HEIGHT h . (IGNORE EFFECTS OF THE CURVATURE OF THE EARTH.)

a) IF THE GUN FIRED NORTH, SHOW THE SHOT IS DEFLECTED

$$\text{BY } d = 2 \sqrt{\frac{2h}{g}} \Omega \left(\frac{4}{3} h \sin \theta - D \cos \theta \right)$$

IN WHAT DIRECTION?

b) IF THE GUN IS FIRED EAST THAT THE SHOT IS DEFLECTED

$$\text{BY } d = 2 \Omega \sqrt{\frac{2h}{g}} D \cos \theta. \quad \text{IN WHAT DIRECTION?}$$

ALSO SHOW THAT IT LANDS A DISTANCE $D' = D \left(1 + \frac{\Omega D \sin \theta}{\sqrt{2gh}} \right)$

TO THE EAST (TO 1ST ORDER IN Ω)