Ph 205 SET 7

DUE TUES. NOV 13, 1990 MAXIMUM RECORDED SCORE = 70 POINTS

(1) IN A SCATTERING EXPERIMENT, THE DIFFERENTIAL CROSS-SECTION IS OBSERVED TO BE

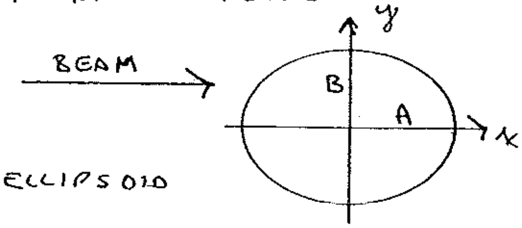
$$\frac{d\sigma}{d\cos\theta} = \frac{\pi a^2}{2}(1+\epsilon\cos\theta) \quad , \epsilon \text{ SMALL}$$

SUPPOSING THE SCATTERING IS ELASTIC SCATTERING OFF A HARD OBJECT, WHAT IS THE SHAPE?

IF $\epsilon = 0$ IT WOULD BE A SPHERE (PROB (8), SET 6)

THE OBJECT IS ALMOST A SPHERE — SAY AN ELLIPSOID

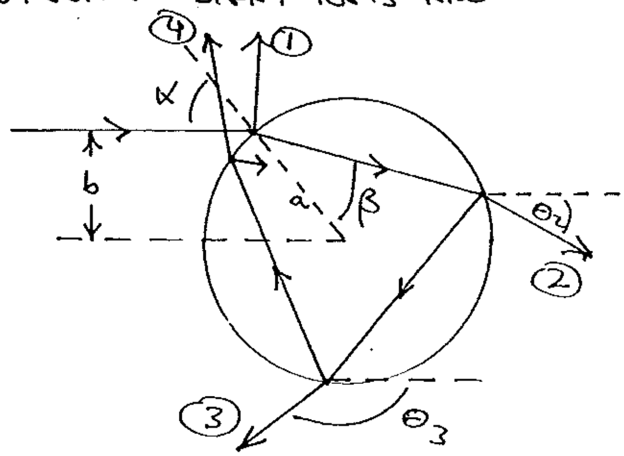
$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{B^2} = 1$$



FIND $\frac{d\sigma}{d\cos\theta}$ FOR SCATTERING OFF AN ELLIPSOID WITH ARBITRARY AXES A AND B. WHAT ARE A AND B CORRESPONDING TO THE CROSS-SECTION STATED ABOVE, SUPPOSING ϵ IS SMALL?

(2) A DIGRESSION INTO OPTICS. [A BOOK: "RAINBOWS, HALOES & GLORIES" BY GREENLER.]

a) RAINBOW SCATTERING. CONSIDER THE SCATTERING OF LIGHT OFF A WATER DROP. WHEN LIGHT HITS A BOUNDARY BETWEEN AIR AND WATER, SOME LIGHT IS TRANSMITTED AND SOME IS REFLECTED. SO MANY OUTGOING LIGHT RAYS ARE POSSIBLE.



THE 1ST 4 OUTGOING RAYS ARE SHOWN IN THE SKETCH.

CASE (1) ~ REFLECTION ⟷ HARD SCATTERING $\Rightarrow$ ISOTROPIC

CASE (2) = PROB (10) SET 6

WE ARE INTERESTED IN CASES (3) AND (4) AS THEY LEAD TO RAINBOWS!

SINCE $b = a\sin\alpha$ AS SHOWN

$$\frac{d\sigma}{d\Omega} = \frac{b\,db}{\sin\theta\,d\theta} = \frac{a^2\sin\alpha\cos\alpha}{\sin\theta}\frac{d\alpha}{d\theta}$$

IF $\frac{d\theta}{d\alpha} = 0$, THEN $\frac{d\sigma}{d\Omega} \to \infty$.

THAT IS, IF MANY DIFFERENT α'S (AND HENCE b'S) LEAD TO THE SAME Θ, THE SCATTERED LIGHT WILL GET VERY BRIGHT — THIS IS THE RAINBOW EFFECT!

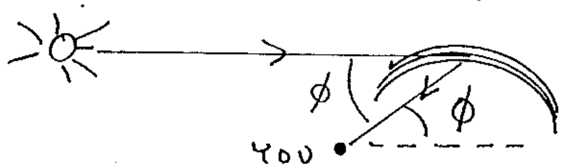
LET m = # OF INTERNAL REFLECTIONS BEFORE THE RAY EMERGES. CALCULATE $\Theta = f(\alpha, \beta, m)$ FROM GEOMETRY.

USE SNELL'S LAW TO RELATE α, β AND THE INDEX OF REFRACTION n.

SHOW THAT $\frac{d\Theta}{d\alpha} = 0$ WHEN $\sin^2\alpha = \frac{(m+1)^2 - n^2}{(m+1)^2 - 1}$

FOR WATER $n \sim 4/3$. EVALUATE α, β & Θ FOR THE FIRST TWO RAINBOWS, $m = 1$ & 2 ($\Theta = 138°, -129°$)

IF YOU ARE WATCHING THE RAINBOW, WHAT IS THE ANGLE BETWEEN THE LIGHT YOU SEE AND THE DIRECTION TO THE SUN?

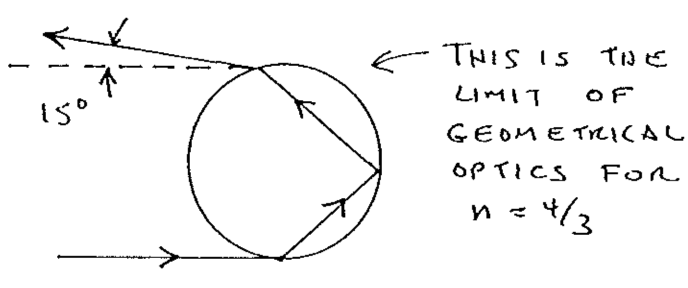


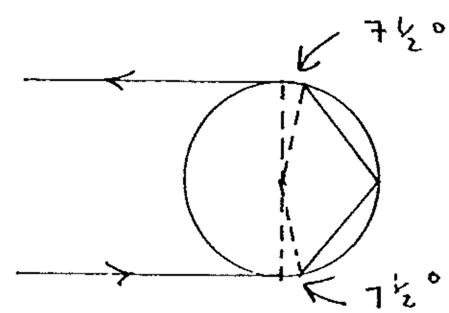
THE INDEX OF REFRACTION, n, VARIES WITH THE WAVELENGTH OF LIGHT. LONG $\lambda \Rightarrow$ SMALL n. WHAT IS THE ORDER OF THE COLORS IN THE 1ST & 2ND RAINBOWS?

THE EXPLANATION OF THE RAINBOW IS ATTRIBUTED TO DESCARTES.

b) <u>GLORIES</u> (STRICTLY CULTURAL, THERE IS NO PROBLEM ASSIGNED)

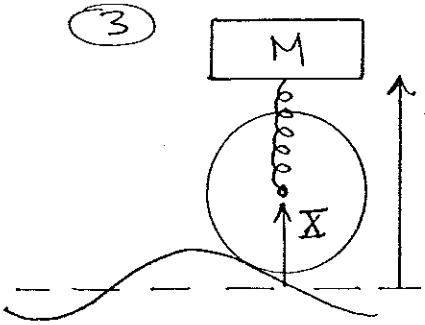
IF YOU LOOK AT THE SHADOW OF THE AIRPLANE ON A CLOUD WHEN YOU ARE FLYING, YOU WILL SEE A HALO OR 'GLORY' OF LIGHT IMMEDIATELY OUTSIDE THE SHADOW. THAT IS, THERE IS AN ENHANCEMENT OF SCATTERING BY 180° OFF WATER DROPS. THIS WAS FIRST OBSERVED IN 1735 BY A SPANISH MOUNTAIN CLIMBER IN THE ANDES. BUT A GOOD EXPLANATION SEEMS TO HAVE BEEN GIVEN ONLY IN 1959.





APPARENTLY SOME LIGHT GETS TRAPPED IN A THIN LAYER AT THE SURFACE OF THE DROP AND IS CARRIED THRU A FEW DEGREES OF ARC BEFORE REFRACTING INTO THE DROP. THE LIGHT WHICH IS CARRIED BY 7½° ON THE WAY IN & ALSO ON THE WAY OUT IS THEN SCATTERED BY 180° AND CAUSES THE GLORY.

(3)



A CAR IS DRIVING ALONG A WASHBOARD ROAD SUCH THAT THE AXLE UNDERGOES A FORCED VERTICAL OSCILLATION:

$$X = A \cos \omega t + X_0$$

MASS M OF THE CAR IS SUPPORTED ABOVE THE AXLE VIA A SHOCK ABSORBER OF REST LENGTH ℓ AND SPRING CONSTANT K. THE DAMPING OF THE SHOCK ABSORBER IS PROPORTIONAL TO THE RATE OF CHANGE OF ITS LENGTH. i.e.,

$$F_{DAMPING} = -b(\dot{x} - \dot{X})$$

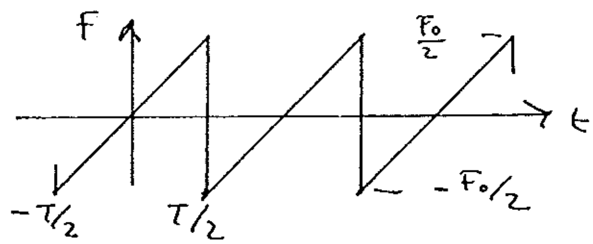
WHERE x = HEIGHT OF TOP OF THE SHOCK ABSORBER ABOVE THE MEAN HEIGHT OF THE ROAD.

WRITE DOWN AND SOLVE THE DIFFERENTIAL EQUATION FOR THE VERTICAL MOTION OF MASS M. SHOW THAT THE AVERAGE HEIGHT IS $\bar{x} = X_0 + \ell - \frac{g}{\omega_0^2}$, AND THE AMPLITUDE OF THE OSCILLATION IS $A\sqrt{\frac{\omega_0^4 + 4\beta^2\omega^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}}$ WHERE $\omega_0^2 = K/M$ AND $2\beta = b/M$

SUPPOSE THE SHOCK ABSORBER IS CRITICALLY DAMPED. AT WHAT FREQUENCY IS THE OSCILLATION AMPLITUDE A MAXIMUM, AND WHAT IS THE MAXIMUM AMPLITUDE? ANS: $AMPLI_{MAX} = \frac{2\sqrt{3}}{3} A$.

(4) a) GIVE THE FOURIER SERIES EXPANSION OF THE SAWTOOTH WAVE FORM

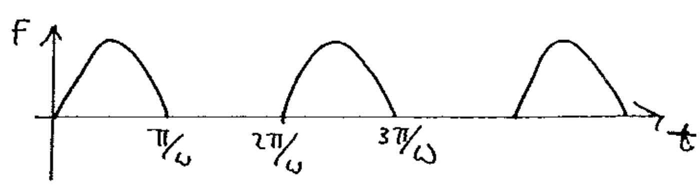
$$F(t) = \frac{F_0 t}{T} \quad \left(-\tfrac{T}{2} < t < \tfrac{T}{2}\right)$$



ANS: $F(t) = \frac{F_0}{\pi}\left\{\sin \omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \dots\right\} \quad \omega \equiv \frac{2\pi}{T}$

b) GIVE THE FOURIER SERIES EXPANSION OF THE HALF-WAVE FUNCTION

$$F(t) = \begin{cases} \sin \omega t & 0 < t < \pi/\omega \\ 0 & \pi/\omega < t < \frac{2\pi}{\omega} \end{cases}$$



ANS: $F(t) = \frac{1}{\pi} + \frac{1}{2}\sin \omega t - \frac{2}{3\pi}\cos 2\omega t - \frac{2}{15\pi}\cos 4\omega t - \dots$

WHICH OF a) OR b) CONVERGES FASTER?

(5) a) A DAMPED OSCILLATOR IS DRIVEN BY A STEP FUNCTION FORCE $F(t) = \begin{cases} 0 & t<0 \\ F_0 & t>0 \end{cases}$

USE GREEN'S METHOD TO CALCULATE THE MOTION $x(t)$

ANS: $x = \frac{F_0}{M\omega_0^2}\left(1 - e^{-\beta t}\cos\omega_1 t - \frac{\beta e^{-\beta t}}{\omega_1}\sin\omega_1 t\right) \qquad t>0$

SKETCH THIS FOR $\beta = 0$, AND $\beta \sim 1/4\,\omega_0$.

NOTE THAT THE OSCILLATION MAKES A LARGE OVERSHOOT OF THE EQUILIBRIUM POSITION $\frac{F_0}{M\omega_0^2}$. WHAT IS THE TIME AT THE MAXIMUM OF THE 1ST OVERSHOOT, AND WHAT IS x THEN?

ANS: $x = \frac{F_0}{M\omega_0^2}\left(1 + e^{-\beta\pi/\omega_1}\right)$

b) THE SAME OSCILLATOR IS NOW SUBJECT TO A FINITE IMPULSE

$$F(t) = \begin{cases} 0 & t<0 \\ F_0 & 0<t<T \\ 0 & t>T \end{cases}$$

NOW WHAT IS $x(t)$?

SUPPOSE THE DAMPING IS STRONG ENOUGH THAT THE INITIAL OSCILLATIONS HAVE DIED OUT BEFORE THE FORCE IS TURNED OFF, I.E. $e^{-\beta T} \sim 0$. SKETCH $x(t)$ IN THIS CASE.

(6) a) A DAMPED OSCILLATOR IS SUBJECT TO THE DRIVING FORCE $F(t) = F_0 e^{-\alpha t}$. SOLVE FOR THE 'STEADY' MOTION $x(t)$ BY MAKING A SUITABLE GUESS AS TO THE FORM OF $x(t)$.

b) NOW SUPPOSE $F(t) = \begin{cases} 0 & t<0 \\ F_0 e^{-\alpha t} & t>0 \end{cases}$

USE GREEN'S METHOD TO SOLVE FOR THE TRANSIENT RESPONSE. (WHICH SHOULD ALSO INCLUDE THE 'STEADY' MOTION OF PART a)!)

ANS: $x = \frac{F_0/M}{\omega_0^2 + \alpha^2 - 2\alpha\beta}\left\{e^{-\alpha t} + e^{-\beta t}\left(\frac{\alpha-\beta}{\omega_1}\sin\omega_1 t - \cos\omega_1 t\right)\right\}$

SKETCH THIS FOR THE CASE $\alpha = \beta$.

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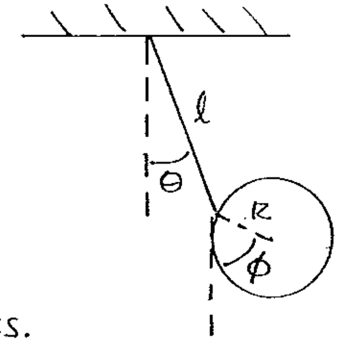
[IN MY OPINION YOU SHOULD WORK AS MANY OF THE LAST 4 PROBLEMS AS YOU CAN.]

(7) A HOOP OF MASS M AND RADIUS R IS ATTACHED TO A MASSLESS ROD OF LENGTH ℓ TO FORM A PENDULUM. THE HOOP PIVOTS FREELY ABOUT THE CONNECTION TO THE ROD. ALL THE MOTION IS IN A VERTICAL PLANE.

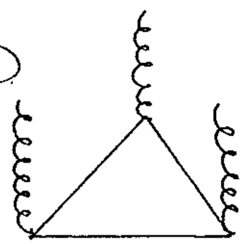


FIND THE FREQUENCIES OF THE NORMAL MODES.

HINT: MAKE THE SMALL-ANGLE APPROXIMATION BEFORE DERIVING THE EQUATIONS OF MOTION — BUT REMEMBER YOU MUST KEEP TERMS TO 2ND ORDER IN T AND V IF YOU USE LAGRANGE'S METHOD.

AS A SPECIAL CASE, SETTING $R = \ell/2$, I GOT $\omega = \sqrt{\frac{g}{\ell}\left(4 \pm 2\sqrt{2}\right)}$

(8)



AN EQUILATERAL TRIANGLE OF MASS M (A THIN PLATE) IS SUSPENDED FROM 3 SPRINGS AT THE 3 CORNERS. THE EQUILIBRIUM POSITION OF THE PLATE IS HORIZONTAL (ALL 3 CORNERS AT THE SAME VERTICAL HEIGHT)

ALL 3 SPRINGS HAVE THE SAME FORCE CONSTANT K, AND THE SAME REST LENGTH.

WHAT ARE THE FREQUENCIES OF THE NORMAL MODES?

(IGNORE ROTATION ABOUT A VERTICAL AXIS, AND CONSIDER ONLY MOTION IN WHICH THE C.M. MOVES VERTICALLY — NO PENDULUM MOTION)

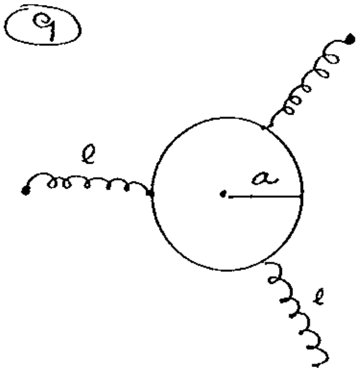
WE ARE LEFT WITH 3 DEGREES OF FREEDOM $\Rightarrow$ 3 MODES. YOU MAY WISH TO WRITE DOWN THE GENERAL EQUATION OF MOTION IN 3 COORDINATES. HOWEVER IT IS SUFFICIENT TO <u>GUESS</u> THE FORM OF THE MOTION OF EACH OF THE 3 MODES. THEN DERIVE AN OSCILLATORY EQUATION FOR ONE MOTION AT A TIME. EACH MODE SHOULD BE DESCRIBED BY ONLY 1 COORDINATE.

SKETCH THE 3 MODES, AND SHOW THAT THE FREQUENCIES ARE

$$\omega_1 = \sqrt{\frac{3K}{M}}, \quad \omega_2 = \omega_3 = 2\omega_1$$

THE FACT THAT $\omega_2 = \omega_3$ MEANS YOU COULD HAVE MADE OTHER DEFINITIONS OF THE MOTION OF THESE 2 MODES. IN PICTURES, WHAT DOES AN ARBITRARY MODE WITH $\omega = 2\omega_1$ LOOK LIKE?

(9)



A UNIFORM DISK OF MASS M, RADIUS a, RESTS ON A SMOOTH TABLE. IT IS CONNECTED VIA 3 SPRINGS OF CONSTANT K, REST LENGTH ℓ_0 TO 3 FIXED POINTS 120° APART. AT EQUILIBRIUM THE SPRINGS HAVE LENGTH $\ell > \ell_0$.

WHAT ARE THE FREQUENCIES OF THE 3 NORMAL MODES (INCLUDING ROTATION)?

YOU MIGHT GUESS THE MODES AND SOLVE THEM ONE BY ONE, OR TRY LAGRANGE'S METHOD.

ANS: $\omega_1 = \omega_2 = \sqrt{\frac{3K}{2M}\,\frac{2\ell-\ell_0}{\ell}}$ $\qquad \omega_3 = \sqrt{\frac{6K}{M}\,\frac{(\ell-\ell_0)(a+\ell)}{a\ell}}$

THE PROBLEMS ON p 60, L&L MAY HELP WITH THE GEOMETRY.

(10) a) CONSIDER THE LINEAR TRIATOMIC MOLECULE OF PROBLEM 1, p 72, L&L. THEY SOLVE IT BY GUESSING THE MODES, & USING THE CONSTANCY OF THE C.M. TO REDUCE THE PROBLEM TO 2 DEGREES OF FREEDOM. (WE WILL IGNORE THE BENDING MODE HERE.) WORK THIS PROBLEM BY DERIVING THE 3 COUPLED EQUATIONS OF MOTION OF COORDS x_1, x_2 & x_3. ASSUME OSCILLATORY SOLUTIONS TO DERIVE THE CHARACTERISTIC EQUATION FOR ω^2. ANS $\omega^2 = 0,\ \frac{K}{M_A},\ K\frac{2M_A+M_B}{M_A M_B}$

THE SOLUTION $\omega = 0$ MEANS THERE IS A NON-OSCILLATORY MOTION POSSIBLE IN THIS SYSTEM — WHICH WE KNOW IS JUST TRANSLATION OF THE C.M.

b) SUPPOSE THE MIDDLE ATOM M_B IS TIED TO THE ORIGIN BY A SPRING, ALSO OF CONSTANT K (REST LENGTH ZERO)

NOW WHAT ARE THE NORMAL FREQUENCIES?

I GOT, $\omega^2 = \frac{K}{M_A},\ K\left\{\frac{3M_A+M_B \pm \sqrt{9M_A^2+2M_AM_B+M_B^2}}{2M_AM_B}\right\}$