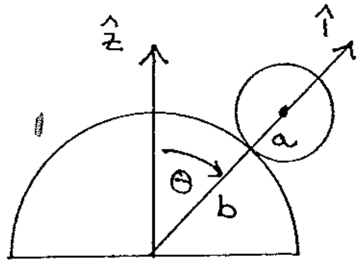
PH 205 SET 11

DUE THURSDAY, DEC 14, 1990 ; MAXIMUM RECORDED SCORE = 60 POINTS

① SPINNING BASKETBALLS



THE HARLEM GLOBETROTTERS CAN BALANCE A BASKETBALL BY SPINNING IT ON A FINGER. WE SEE THAT STABILITY MIGHT BE POSSIBLE IF THE SPINNING BALL ACTS LIKE A GYROSCOPE AND PRECESSES, RATHER THAN ROLLS OFF.

CONSIDER A SPHERE OF RADIUS a, MASS M WHICH ROLLS WITHOUT SLIPPING ON A FIXED SPHERE OF RADIUS b. DERIVE AND DECOMPOSE INTO COMPONENTS THE EQUATION OF MOTION. SOME MILESTONES:

$$\bar{\omega} = \omega_1 \hat{\imath} + \frac{a+b}{a} \hat{\imath} \times \frac{d\hat{\imath}}{dt}$$

$$(I+ma^2)\frac{a+b}{a}\hat{\imath} \times \frac{d^2\hat{\imath}}{dt^2} + I\omega_1 \frac{d\hat{\imath}}{dt} + mga\,\hat{\imath}\times\hat{z} = 0$$

NOTE THAT $\hat{\imath}$ ROTATES ABOUT $\hat{z}$ WITH RATE $\dot{\phi}$, AND ABOUT $\hat{\imath}\times\hat{z}$ AT RATE $\dot{\theta}$ (CAREFUL OF SIGNS).

AFTER OBTAINING THE 3 COMPONENT EQUATIONS OF MOTION, FIRST CONSIDER THE STEADY SOLUTION, $\dot{\theta}=0$, $\dot{\phi}=\Omega$ TO SHOW THAT THE ANGULAR VELOCITY ALONG THE $\hat{\imath}$ AXIS MUST OBEY

$$\omega_1 > \frac{2}{I}\sqrt{mg(a+b)(I+ma^2)\cos\theta}$$

THE SPHERE WILL FALL OFF IF THE RADIAL COMPONENT OF THE CONTACT FORCE VANISHES. SHOW THAT THIS REQUIRES

$$\Omega^2 < \frac{g\cos\theta}{(a+b)\sin^2\theta}$$

USE THE RELATION BETWEEN Ω AND ω_1 TO SHOW THAT THIS INDICATES THAT TOO MUCH SPIN IS BAD AS WELL AS TOO LITTLE.

CONSIDER NUTATIONS ABOUT STEADY PRECESSION

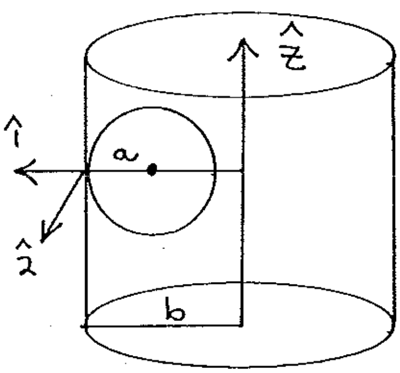
$$\theta = \theta_0 + \epsilon\sin\alpha t$$
$$\dot{\phi} = \Omega + \delta\sin\alpha t$$

TO SHOW $$\alpha^2 = \frac{I^2\omega_1^2 - 4mg(a+b)(I+ma^2)\cos\theta}{\left[(I+ma^2)\left(\frac{b+a}{a}\right)\right]^2} + \Omega^2$$

SO INDEED SMALL OSCILLATIONS ARE STABLE IF $\omega_1 > \omega_{1\,MIN}$ FOUND ABOVE.

FOR A BASKETBALL OF RADIUS 15 cm WHICH IS A HOLLOW SPHERE, $I = \frac{2}{3} ma^2$, BALANCED VERTICALLY ON YOUR FINGER, $b \sim 1$ cm, OUR RESULT PREDICTS A MINIMUM ANGULAR FREQUENCY OF ~ 80 Hz FOR STABILITY. THIS IS RATHER FAST & WE RELUCTANTLY CONCLUDE THAT THE HARLEM GLOBETROTTERS NEVER TOOK Ph 205.

② THE GOLFER'S NEMESIS CAN A GOLF BALL ROLL INTO THE CUP, ROLL AROUND ON THE VERTICAL WALL & THEN POP BACK OUT?



CONSIDER A SPHERE OF RADIUS a, MASS M, ROLLING WITHOUT SLIPPING INSIDE A VERTICAL CYLINDER OF RADIUS b.

IF $\Omega = \dot{\phi}$ = ROTATION OF THE POINT OF CONTACT ABOUT THE VERTICAL, SHOW THAT THE COMPONENTS OF THE EQUATION OF MOTION LEAD TO

$$\uparrow: \quad a\dot{\omega}_1 = \Omega \dot{z}$$

$$\hat{2}: \quad (I + ma^2)\ddot{z} = -ma^2 g - I a \omega_1 \Omega$$

$$\hat{z}: \quad \dot{\Omega} = 0$$

SHOW THAT z OF C.M. EXECUTES SIMPLE HARMONIC MOTION, AND IF AT $t=0$, $z=0$, $\dot{z}=0$ AND $\omega_1 = \omega_{10}$, THEN

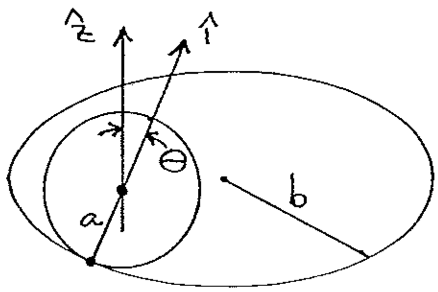
$$z = \left(\frac{ma^2 g + I a \Omega \omega_{10}}{I \Omega^2}\right)(\cos \omega_z t - 1)$$

WHERE $\omega_z = \Omega \sqrt{\dfrac{I}{I + ma^2}}$

NOTE THAT IF THE BALL ROLLS INTO THE CUP WITH VELOCITY v_0, THEN $\Omega = \dfrac{v_0}{b-a}$ (IF CONDITIONS ARE RIGHT FOR ROLLING WITHOUT SLIPPING...)

FOR A UNIFORM SPHERE SHOW THAT THE BALL RISES AGAIN TO THE RIM OF THE CUP AFTER 1.87 REVOLUTIONS, AND SO INDEED MIGHT POP OUT!

③ OFF THE RIM



A FREQUENT OCCURENCE IN GOLF OR BASKETBALL IS THAT THE BALL ROLLS AROUND THE RIM OF THE CUP OR BASKET FOR QUITE A WHILE — THEN SOMETIMES GOES IN, SOMETIMES NOT...

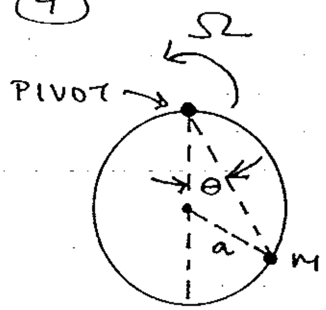
CONSIDER A SPHERE OF RADIUS a, MASS M, ROLLING WITHOUT SLIPPING ON A HORIZONTAL HOOP OF RADIUS b. AN EQUILIBRIUM OF STEADY ROLLING EXISTS WITH NO 'SPIN' COMPONENT, $\omega_1 = \vec{\omega}\cdot\hat{1} = 0$. IN THIS CASE SHOW THAT THE ANGULAR VELOCITY OF THE POINT OF CONTACT ABOUT THE VERTICAL IS

$$\Omega = \sqrt{\frac{3g\tan\Theta}{5(b - a\sin\Theta)}} \qquad \text{FOR A } \underline{\text{HOLLOW}} \text{ SPHERE.}$$

FOR A BASKET BALL OF RADIUS 15 CM, A HOOP OF RADIUS 30 CM, THIS GIVES $\sim$ 1 REV/SEC. AT $\Theta = 45°$.

SHOW THAT THE EQUILIBRIUM IS UNSTABLE. IF $\Omega > \Omega_{EQUI}$, THEN THE BALL RISES AND WILL LEAVE THE HOOP. IF $\Omega < \Omega_{EQUI}$, THEN THE BALL WILL FALL THRU AS DESIRED.

④



A CIRCULAR HOOP OF RADIUS a ROTATES WITH CONSTANT ANGULAR VELOCITY Ω IN A HORIZONTAL PLANE. THE PIVOT IS A POINT ON THE HOOP. A BEAD OF MASS M SLIDES FREELY ON THE HOOP.

a) USE Θ AS SHOWN AND LAGRANGE'S METHOD TO SHOW THAT THE EQUATION OF MOTION IS

$$\ddot{\Theta} = -\Omega^2 \sin\Theta\cos\Theta$$

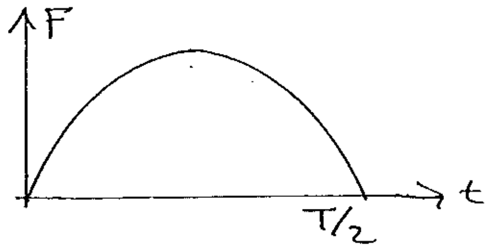
b) SHOW THAT THE HAMILTONIAN IS $H = \frac{P_\Theta^2}{8Ma^2} - P_\Theta\Omega\cos^2\Theta - \frac{Ma^2}{2}\Omega^2\sin^2 2\Theta$

AND THAT HAMILTON'S EQUATIONS ALSO LEAD TO THE RESULT OF a). IS ENERGY CONSERVED?

c) ANALYZE THE PROBLEM IN A ROTATING FRAME. WITH CARE THE RESULT OF a) FOLLOWS FAIRLY QUICKLY.

(5) THE PIANO A PIANO WIRE IS STRUCK BY A HAMMER WITH A SHARP BLOW, AND A FAIRLY PURE NOTE IS PRODUCED. THIS IS SURPRISING GIVEN THE ANALYSIS IN THE NOTES OF THE EFFECT OF AN IMPULSE. HELMHOLTZ HAS SUGGESTED THAT A BETTER APPROXIMATION TO THE EFFECT OF THE HAMMER IS

$$F(x,t) = \begin{cases} F\,\delta(x-b)\sin\dfrac{2\pi t}{T} & 0<t<\dfrac{T}{2} \\ 0 & \text{OTHERWISE} \end{cases}$$



I.e. THE FORCE GOES THRU ONE HALF-PERIOD OF A HARMONIC OSCILLATION.

THE FORCE IS APPLIED AT A POINT b FROM THE END OF THE WIRE OF LENGTH ℓ. THE WIRE IS FIXED AT BOTH ENDS AND STRETCHED SO THAT THE WAVE VELOCITY IS c.

USE GREEN'S METHOD TO SHOW THAT THE STRING VIBRATES AS

$$S(x,t) = \frac{2FT}{\pi^2 c\rho} \sum_n \frac{\sin\frac{n\pi b}{\ell}\cos\frac{n\pi cT}{4\ell}}{n\left(1-\left(\frac{ncT}{2\ell}\right)^2\right)} \sin n\pi x \sin\frac{n\pi c}{\ell}\left(t-\frac{T}{4}\right)$$

SUPPOSE WE CHOOSE $b=\ell/2$, THE MIDPOINT,

AND $T=\dfrac{2\ell}{c}$ = FUNDAMENTAL PERIOD

THEN $$S(x,t) = \frac{2F\ell}{\pi^2 T}\sum_n \frac{\sin n\pi}{n(1-n^2)}\sin\frac{n\pi x}{\ell}\sin\frac{n\pi c}{\ell}\left(t-\frac{\ell}{2c}\right)$$

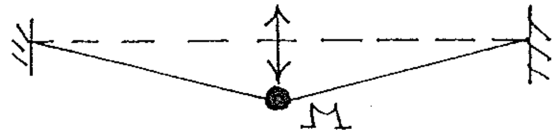
SO ALL HARMONICS VANISH EXCEPT $n=1$

$$\left(\text{SINCE } \lim_{n\to 1}\frac{\sin n\pi}{1-n^2} = \frac{n\cos n\pi}{-2n} = \frac{1}{2}\right)$$

EVEN IF $T=\dfrac{2\ell}{c}$ CANNOT BE EXACTLY ACHIEVED IN PRACTICE, THE SERIES CONVERGES QUICKLY SINCE THE TERMS GO LIKE $1/n^3$.

(6) A STRING OF MASS m, LENGTH ℓ IS CLAMPED AT BOTH ENDS AND STRETCHED WITH A TENSION T.

a) A MASS M IS ATTACHED TO THE MIDPOINT



IGNORE THE MASS OF THE STRING AND SHOW THAT MASS M OSCILLATES TRANSVERSELY WITH FREQUENCY

$$\Omega_0 = 2\sqrt{\frac{T}{M\ell}}$$

b) SUPPOSE THE MASS IS ATTACHED AT A DISTANCE b FROM ONE END. LET $c = \sqrt{T/\rho}$ = VELOCITY ALONG STRING.

USE THE METHOD OF DIVIDING THE PROBLEM INTO TWO STRINGS OVER INTERVALS $[0,b]$ AND $[b,\ell]$. SHOW THAT THE NORMAL FREQUENCIES OBEY THE TRANSCENDENTAL EQUATION

$$\Omega \sin\frac{\Omega b}{c} \sin\Omega\frac{\ell-b}{c} = \frac{T}{Mc}\sin\frac{\Omega\ell}{c}$$

c) CONSIDER AGAIN THE CASE $b = \ell/2$.

SHOW THAT THERE ARE 2 CLASSES OF SOLUTIONS:

1) $\Omega = \dfrac{2n\pi c}{\ell}$ IN WHICH M DOESN'T MOVE AT ALL

2) M MOVES AND $\left(\dfrac{\Omega\ell}{2c}\right)\tan\left(\dfrac{\Omega\ell}{2c}\right) = \dfrac{\rho\ell}{M} = \dfrac{m}{M}$

d) IF $M \ll m$ SHOW THAT THE LOWEST FREQUENCY IS

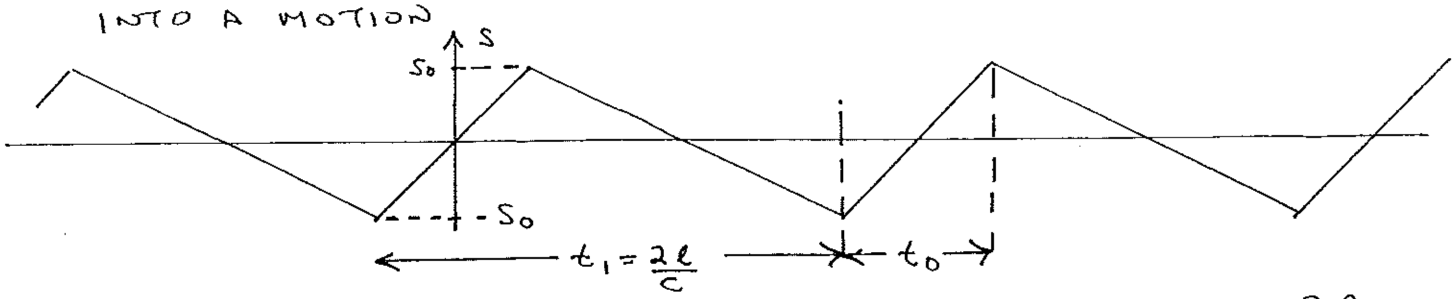
$$\Omega \sim \Omega_1\left(1 - \frac{M}{m}\right) \quad \text{WHERE } \Omega_1 = \frac{\pi c}{\ell} = \text{FREQUENCY IF } M = 0$$

e) IF $\frac{m}{M} \ll 1$, KEEP ENOUGH HIGHER ORDER TERMS TO SHOW THAT THE LOWEST FREQUENCY IS

$$\Omega \sim \Omega_0\left(1 - \frac{m}{6M}\right) \Longleftrightarrow \Omega = 2\sqrt{\frac{T}{\ell(M + m/3)}}$$

SO THAT THE MASS OF THE STRING APPEARS AS A CORRECTION $m/3$ TO THE HEAVY MASS M.

(7) THE VIOLIN BY MEANS OF EXPERIMENT, HELMHOLTZ DEDUCED THAT THE ACTION OF A VIOLIN BOW ON A STRING IS TO FORCE THE POINT OF CONTACT OF THE STRING INTO A MOTION



WHICH IS PERIODIC WITH THE PERIOD OF THE 1ST HARMONIC $t_1 = \frac{2\ell}{c}$

IF THE POINT OF APPLICATION OF THE BOW IS x_0, THE RISING MOTION OCCUPIES A TIME t_0 RELATED BY $\frac{x_0}{\ell} = \frac{t_0}{t_1}$.

FIRST MAKE A FOURIER ANALYSIS IN TIME OF THE MOTION OF THE POINT OF CONTACT TO SHOW

$$S(x_0, t) = \frac{2 S_0 t_1^2}{\pi^2 t_0 (t_1 - t_0)} \cdot \sum_n \frac{1}{n^2} \sin \frac{n\pi x_0}{\ell} \sin \frac{2n\pi t}{t_1}$$

IN GENERAL WE EXPECT THE MOTION OF THE ENTIRE STRING TO BE ANALYZABLE AS

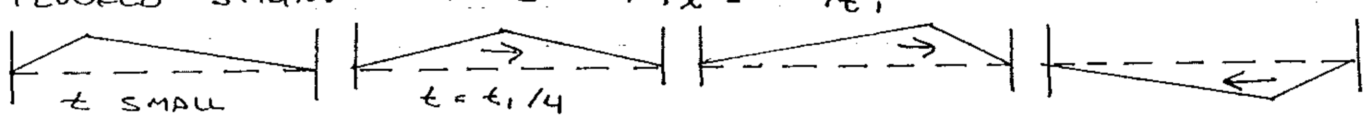
$$S(x,t) = \sum_n \sin \frac{n\pi x}{\ell} \left(A_n \cos \frac{2n\pi t}{t_1} + B_n \sin \frac{2n\pi t}{t_1} \right)$$

HENCE $A_n = 0$ AT ONCE, AND

$$S(x,t) = \frac{2 S_0 t_1^2}{\pi^2 t_0 (t_1 - t_0)} \sum_n \frac{1}{n^2} \sin \frac{n\pi x}{\ell} \sin \frac{2n\pi t}{t_1}$$

FROM THE NOTES, WE SAW THAT AT $t = 0$, A PLUCKED STRING HAS THE ANALYSIS $\frac{2 S_0 \ell^2}{\pi^2 b (\ell - b)} \sum_n \frac{1}{n^2} \sin \frac{n\pi x}{\ell} \sin \frac{n\pi b}{\ell}$

HENCE AT ANY MOMENT THE VIOLIN STRING LOOKS LIKE A PLUCKED STRING WHERE $b/\ell = 2t/t_1$



THE CREST OF THE MOTION MOVES ALONG THE STRING WITH VELOCITY $c = 2\ell/t_1$. THE 'VIBRATION' IS MORE LIKE A TRAVELLING WAVE THAN A STANDING WAVE!

(8) A SPRING OF REST LENGTH l_0 HAS MASS m.

ONE END IS FIXED AND THE OTHER HAS A MASS M ATTACHED TO IT.

SET UP THE BOUNDARY CONDITIONS, AND SOLVE THE WAVE EQUATION FOR THE NORMAL FREQUENCIES OF LONGITUDINAL OSCILLATION. (IGNORE GRAVITY) TAKE THE SPRING AS A UNIFORM BAR....

SHOW $\cot(\Omega l_0) = \frac{M}{m}(\Omega l_0)$

WHERE $\Omega = \sqrt{\frac{m\omega^2}{K l_0^2}}$

K = SPRING CONSTANT

ω = OSCILLATION FREQUENCY

BY SUITABLE APPROXIMATION, SHOW THAT THE FREQUENCY OF THE LOWEST MODE IS

$$\omega \sim \sqrt{\frac{K}{M + m/3}}$$

AS DERIVED ON PROBLEM SET 1.