Ph 205 LECTURE 4

TWO EXAMPLES OF THE USE OF LAGRANGE'S METHOD

EXAMPLE 1 MOTION OF A SINGLE PARTICLE.

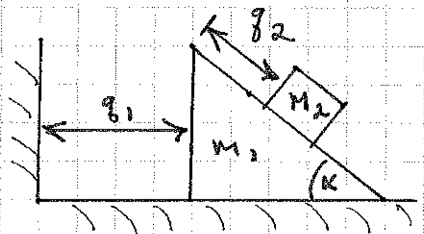
WE CHOOSE (x, y, z) AS THE 'GENERALIZED COORDINATES'

$$L = T - V = \tfrac{1}{2}mv^2 - V(r) = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - V\left(\sqrt{x^2+y^2+z^2}\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} \qquad \frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = F_x$$

OR $\bar{F} = m\bar{a}$ AND WE'RE BACK TO NEWTON!

EXAMPLE 2 THE BLOCK SLIDES ON THE WEDGE WHICH SLIDES ON THE PLANE. THERE IS NO FRICTION.



THERE ARE SEVERAL CONSTRAINTS IN THIS PROBLEM. LAGRANGE ADVISES US TO ASK AT ONCE — HOW MANY COORDINATES ARE NEEDED?

THE ANSWER IS <u>TWO</u>! A SUITABLE CHOICE IS:

q_1 = HORIZONTAL POSITION OF THE WEDGE.
q_2 = DISTANCE BLOCK 2 HAS SLID DOWN THE WEDGE

THAT'S ALL!

THE COORDINATE TRANSFORMATION EQUATIONS ARE

$$x_1 = q_1 \qquad x_2 = q_1 + q_2\cos\alpha$$
$$y_1 = \text{const} \qquad y_2 = h - q_2\sin\alpha$$

THEN $T_1 = \frac{1}{2}m_1\dot{q}_1^2$, $T_2 = \frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2) = \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha + \dot{q}_2^2)$

$$V = m_2 g y_2 = -m_2 g q_2 \sin\alpha \qquad \text{SETTING } h = 0.$$

$$L = T - V = \tfrac{1}{2}(m_1+m_2)\dot{q}_1^2 + m_2\dot{q}_1\dot{q}_2\cos\alpha + \tfrac{1}{2}m_2\dot{q}_2^2 + m_2 g q_2 \sin\alpha$$

$$\frac{\partial L}{\partial \dot{q}_1} = (m_1+m_2)\dot{q}_1 + m_2\cos\alpha\,\dot{q}_2 \; ; \; \frac{\partial L}{\partial q_1} = 0$$

$$\therefore \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = 0 = (m_1+m_2)\ddot{q}_1 + m_2\cos\alpha\,\ddot{q}_2 \Rightarrow \ddot{q}_2 = -\frac{(m_1+m_2)}{m_2\cos\alpha}\ddot{q}_1$$

$$\frac{\partial L}{\partial \dot{q}_2} = m_2\cos\alpha\,\dot{q}_1 + m_2\dot{q}_2 \qquad \frac{\partial L}{\partial q_2} = m_2 g\sin\alpha$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = m_2\cos\alpha\,\ddot{q}_1 + m_2\ddot{q}_2 = m_2 g\sin\alpha$$

$$\ddot{x}_1 = \ddot{q}_1 = \frac{-m_2 g \sin\alpha\cos\alpha}{m_1 + m_2 \sin^2\alpha} \qquad \ddot{q}_2 = \frac{(m_1+m_2) g \sin\alpha}{m_1 + m_2 \sin^2\alpha}$$

$$\ddot{x}_2 = \ddot{q}_1 + \ddot{q}_2 \cos\alpha = \frac{m_1 g \sin\alpha\cos\alpha}{m_1 + m_2 \sin^2\alpha}$$

$$\ddot{y}_2 = -\ddot{q}_2 \sin\alpha = \frac{-(m_1+m_2) g \sin^2\alpha}{m_1 + m_2 \sin^2\alpha}$$

[CHECK THE SPECIAL CASES OF $\alpha = 0°$ OR $90°$]

AS BLOCK 2 FALLS, ITS C.M. MOVES WITH ANGLE $\tan\theta = \frac{\ddot{y}_2}{\ddot{x}_2} = \frac{m_1+m_2}{m_1}\tan\alpha$ TO THE HORIZONTAL.

REMEMBER, LAGRANGE'S GOAL WAS TO REDUCE MECHANICS TO A MATHEMATICAL PROCEDURE.

ANOTHER DERIVATION OF LAGRANGE'S EQUATIONS

(SEE SEC. 2-7 OF BARGER & OLSSON)

OUR GOAL IS TO REPLACE THE LARGE NUMBER OF NEWTONIAN EQUATIONS OF MOTION $\frac{d\bar{P}_i}{dt} = \bar{F}_i$ BY A MINIMAL SET OF EQUATIONS IN A MORE USEFUL SET OF INDEPENDENT GENERALISED COORDINATES q_j.

CAN WE FIND A KIND OF 'GENERALISED MOMENTUM', P_j SUCH THAT $\frac{d P_j}{dt} = Q_j =$ GENERALISED FORCE ?

WE HAVE ALREADY DEFINED THE GENERALISED FORCE

$$Q_j = \sum_i \bar{F}_i^e \cdot \frac{\partial \bar{r}_i}{\partial q_j} \qquad \text{SUCH THAT } \delta W = \sum_j Q_j \delta q_j = \sum_i \bar{F}_i^e \cdot \delta\bar{r}_i$$

WE CANNOT WRITE $P_j = m_j \dot{q}_j$ SINCE m_j DOES NOT MEAN ANYTHING — THE GENERALISED COORDINATES ARE NOT NECESSARILY ASSOCIATED WITH INDIVIDUAL MASSES.

WE TAKE A HINT FROM THE RELATION

$$T = \sum_i \tfrac{1}{2} m_i v_i^2 = \sum_i \tfrac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

SO $P_{i_x} = \frac{\partial T}{\partial \dot{x}_i}$ ETC.

IN TERMS OF THE GENERALISED COORDINATES, T HAS THE SAME VALUE, BUT NOW WE WRITE $T = T(q_j, \dot{q}_j, t)$ USING

$$\bar{v}_i = \frac{d\bar{r}_i}{dt} = \frac{\partial \bar{r}_i}{\partial t} + \sum_j \frac{\partial \bar{r}_i}{\partial q_j}\dot{q}_j \quad \text{AND} \quad \bar{r}_i = \bar{r}_i(q_j, t)$$

BY ANALOGY WE MAKE THE DEFINITION

$$\underline{P_j = \frac{\partial T}{\partial \dot{q}_j}} = \underline{\text{GENERALISED MOMENTUM}}$$

IT MAY NOT HAVE THE DIMENSIONS OF MOMENTUM.

WE CAN RELATE IT TO THE INDIVIDUAL PARTICLE MOMENTA $\bar{P}_i$

$$P_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i \frac{\partial T}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \dot{q}_j} + \frac{\partial T}{\partial \dot{y}_i}\frac{\partial \dot{y}_i}{\partial \dot{q}_j} + \frac{\partial T}{\partial \dot{z}_i}\frac{\partial \dot{z}_i}{\partial \dot{q}_j} = \sum_i \bar{P}_i \cdot \frac{\partial \bar{v}_i}{\partial \dot{q}_i}$$

$$\text{SO} \quad P_j = \sum_i \bar{P}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} \qquad \text{SIMILAR TO} \quad Q_j = \sum_i \bar{F}_i^e \cdot \frac{\partial \bar{r}_i}{\partial q_j}$$

THE EQUATION OF MOTION IS THEN

$$\underline{\frac{dP_j}{dt}} = \frac{d}{dt}\sum_i \bar{P}_i \cdot \frac{\partial \bar{r}_i}{\partial q_j} = \sum_i \frac{d\bar{P}_i}{dt}\cdot\frac{\partial \bar{r}_i}{\partial q_j} + \sum_i \bar{P}_i \cdot \frac{d}{dt}\frac{\partial \bar{r}_i}{\partial q_j}$$

$$= \sum_i \bar{F}_i \cdot \frac{\partial \bar{r}_i}{\partial q_i} + \sum_i m_i \bar{v}_i \cdot \frac{\partial \bar{v}_i}{\partial q_j} = \underline{Q_j + \frac{\partial T}{\partial q_j}}$$

WE USED THE FACT THAT $\sum_i \bar{F}_i^c \cdot \frac{\partial \bar{r}_i}{\partial q_j} = 0$ BY THE STRONG FORM OF THE PRINCIPLE OF VIRTUAL WORK: $0 = \sum_i \bar{F}_i^c \,\delta\bar{r} = \sum_j\sum_i \bar{F}_i^c \cdot \frac{\partial \bar{r}_i}{\partial q_j}\,\delta q_j$

WE DID NOT SUCCEED IN ESTABLISHING THAT $\frac{dP_j}{dt} = Q_j$,

BUT WE FIND AN ADDITIONAL TERM $\frac{\partial T}{\partial q_j}$. THIS IS A KIND OF <u>FICTITIOUS</u> <u>FORCE</u> WHICH APPEARS BECAUSE OF OUR CHOICE OF COORDINATE SYSTEM. FOR EXAMPLE, THE CENTRIFUGAL FORCE EXPERIENCED IN A ROTATING COORDINATE SYSTEM. THAT SUCH TERMS APPEAR AUTOMATICALLY IS A TESTAMENT TO THE POWER OF LAGRANGE'S METHOD.

AS BEFORE, IF THE EXTERNAL FORCES ARE CONSERVATIVE,

$$Q_j = -\frac{\partial V}{\partial q_j} \quad \text{AND} \quad \frac{\partial V}{\partial \dot{q}_j} = 0$$

SO $$\frac{d P_j}{dt} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = \frac{\partial (T-V)}{\partial q_j}$$

AGAIN WE WRITE $L = T - V$ AND SO

$$\boxed{\frac{d P_j}{dt} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} \quad ; \quad P_j = \frac{\partial L}{\partial \dot{q}_j}}$$

LAGRANGE'S EQUATIONS AGAIN.

NOTE THAT IF $\frac{\partial L}{\partial q_j} = 0$ WE HAVE A <u>GENERALISED</u> <u>CONSERVATION</u> <u>LAW</u>,

P_j = CONSTANT.

Four More Examples

<u>EXAMPLE 1</u> MOTION IN A PLANE UNDER A CENTRAL FORCE.

INSTEAD OF x AND y, WE CHOOSE r AND ϕ AS OUR GENERALISED COORDINATES.

$$T = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m (\dot{r}^2 + r^2\dot{\phi}^2) \quad ; \quad V = V(r) \quad ; \quad L = T - V$$

$P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ = RADIAL MOMENTUM. $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$

$$\frac{\partial L}{\partial r} = m r \dot{\phi}^2 - \frac{\partial V}{\partial r} = m r \dot{\phi}^2 + F$$

THE RADIAL EQUATION OF MOTION IS THEN $m\ddot{r} = F + m r \dot{\phi}^2$

THE EXTRA PIECE $m r \dot{\phi}^2$ OCCURS BECAUSE $\frac{\partial T}{\partial r} \neq 0$. THIS IS THE "CENTRIFUGAL FORCE", ARISING DUE TO OUR CHOICE OF COORDINATES.

TURNING TO ϕ, $P_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi}$ = <u>ANGULAR</u> MOMENTUM

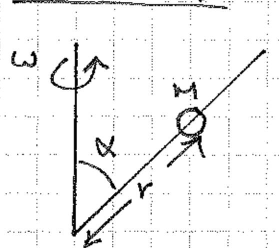
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = m r^2 \ddot{\phi} + 2 m r \dot{r} \dot{\phi} \quad ; \quad \frac{\partial L}{\partial \phi} = 0 \Rightarrow r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0$$

ALSO SINCE $\partial L/\partial \phi = 0$ WE HAVE A CONSERVATION LAW, P_ϕ = CONSTANT.

$\Rightarrow$ ANGULAR MOMENTUM IS CONSERVED IN CENTRAL FORCE PROBLEMS.

IT IS REMARKABLE HOW LAGRANGE'S METHOD FORCES US TO TAKE NOTE OF THIS ESSENTIAL PROPERTY OF THE MOTION.

EXAMPLE 2



A BEAD OF MASS m SLIDES ON A STRAIGHT WIRE WHICH MAKES ANGLE α TO THE VERTICAL. THE WHOLE SYSTEM IS FORCED TO ROTATE ABOUT THE VERTICAL WITH CONSTANT ANGULAR VELOCITY ω.

FOR WHAT VALUE OF r WILL THE BEAD REMAIN FIXED RELATIVE TO THE WIRE?

THERE IS ONLY ONE COORDINATE REQUIRED — r

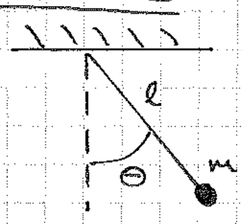
$$T = \tfrac{1}{2} m\dot{r}^2 + \tfrac{1}{2} m(\omega r \sin\alpha)^2, \quad V = mgr\cos\alpha, \quad L = T - V$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r} = \frac{\partial L}{\partial r} = m\omega^2 r\sin^2\alpha - mg\cos\alpha$$

AT EQUILIBRIUM $\ddot{r} = 0 \Rightarrow r = \dfrac{g\cos\alpha}{\omega^2\sin^2\alpha}$

[CAN YOU DO THIS USING CENTRIFUGAL FORCE?]

EXAMPLE 3



THE SIMPLE PENDULUM.

AS THE ONE GENERALISED COORDINATE REQUIRED, WE CHOOSE THE ANGLE θ.

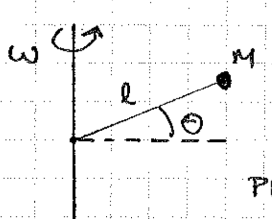
$$T = \tfrac{1}{2} m(l\dot{\theta})^2 \quad ; \quad V = mgl(1-\cos\theta)$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} = ml^2\dot{\theta} = \text{ANGULAR MOMENTUM}$$

THE GENERALISED FORCE IS $Q_\theta = -\dfrac{\partial V}{\partial \theta} = \dfrac{\partial L}{\partial \theta} = -mgl\sin\theta$ = TORQUE

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \Rightarrow ml^2\ddot{\theta} = -mgl\sin\theta \quad \text{OR} \quad \ddot{\theta} = -\frac{g}{l}\sin\theta$$

EXAMPLE 4



PROBLEM 10 d, SET 1.

MASS m IS AT THE END OF A MASSLESS ROD OF LENGTH l WHICH IS ATTACHED TO A VERTICAL AXLE. THE ROD IS FREE TO ROTATE IN A VERTICAL PLANE, BUT THIS PLANE IS FORCED TO ROTATE ABOUT THE VERTICAL AXIS WITH ANGULAR VELOCITY ω. DESCRIBE THE MOTION, NEGLECTING GRAVITY. DESPITE THE COMPLEXITY, WE NEED ONLY ONE COORDINATE, θ AS SHOWN. THEN $T = \tfrac{1}{2} m(\omega l\cos\theta)^2 + \tfrac{1}{2} m(l\dot{\theta})^2 \quad V = 0$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta} = \frac{\partial L}{\partial \theta} = -m\omega^2 l^2\cos\theta\sin\theta$$

SO $\quad \ddot{\theta} = -\omega^2\sin\theta\cos\theta$

IN THE SMALL ANGLE APPROXIMATION, $\ddot{\theta} \approx -\omega\theta$

SO $\quad \theta(t) = \theta_{MAX}\cos\omega t \quad \ldots$

REMARKS ON PROBLEM SOLVING

A MAJOR GOAL OF OUR STUDY OF LAGRANGE'S METHOD IS TO BECOME BETTER PROBLEM SOLVERS. HOWEVER, A HIGH-POWERED COOKBOOK APPROACH IS NO SUBSTITUTE FOR A BASIC FEELING FOR THE PHYSICS OF THE PROBLEM. LAGRANGE'S METHOD SERVES TO EMPHASIZE THE IMPORTANCE OF CERTAIN PROCEDURES THAT ARISE NATURALLY IN THE ELEMENTARY USE OF NEWTON'S LAWS. NONETHELESS THE SPIRIT OF THE ELEMENTARY APPROACH REMAINS A USEFUL GUIDE TO PROBLEM SOLVING.

FOR ME, SOME STEPS IN TYPICAL PROBLEM SOLVING INCLUDE:

1) UNDERSTAND THE PROBLEM BEFORE ATTEMPTING TO WRITE DOWN A SOLUTION. DRAW A FIGURE, LAGRANGE NOTWITHSTANDING. TRY TO EXPLAIN THE NATURE OF THE SOLUTION WITHOUT USING ANY MATHEMATICS. LOOK FOR LIMITING, OR SPECIAL CASES OF THE SAME PROBLEM. LOOK FOR MORE THAN ONE POINT OF VIEW.

2) IDENTIFY THE FORCES, AND THE CONSTRAINTS ON THE POSSIBLE MOTION. LAGRANGE TELLS US THAT IN MANY CASES IT IS MORE IMPORTANT TO UNDERSTAND THE INTERNAL CONSTRAINTS THAN THE INTERNAL FORCES.

3) CHOOSE A GOOD SET OF VARIABLES TO DESCRIBE THE MOTION. AGAIN LAGRANGE REVEALS THAT MUCH OF THE ART OF PROBLEM SOLVING IS IN THE HAPPY CHOICE OF INDEPENDENT (GENERALISED) COORDINATES. DECIDE HOW MANY DEGREES OF FREEDOM THE PROBLEM HAS. IF YOU ARE USING MORE COORDINATES THAN THE NUMBER OF DEGREES OF FREEDOM, YOU MAY BE WORKING TOO HARD.

4) LOOK FOR CONSERVED QUANTITIES. IT MAY NOT BE NECESSARY TO USE THE DIFFERENTIAL EQUATIONS OF MOTION AT ALL.

5) WRITE DOWN THE EQUATIONS OF MOTION. LAGRANGE'S METHOD MAY HELP IN DIFFICULT CASES.

6) SOLVE THE EQUATIONS OF MOTION. THIS IS STRICTLY MATHEMATICS.

7) INTERPRET THE RESULT. OUR GOAL IS NOT MATHEMATICAL MANIPULATION, BUT PHYSICAL INSIGHT INTO THE PROBLEM. THE MATHEMATICS MAY HAVE TOLD US MORE THAN EXPECTED, SO EXPLORE THE FEATURES OF THE RESULT BEFORE QUITTING.

8) DO IT MORE THAN ONE WAY. WHEN YOU SOLVE A PROBLEM NO ONE HAS EVER SOLVED BEFORE, YOU CAN'T CHECK THE ANSWER IN THE BACK OF THE BOOK!

THE REAL DIFFICULTY IN PROBLEM SOLVING IS COMING UP WITH A SOLVABLE PROBLEM IN THE FIRST PLACE!

SOLVING FOR THE $\ddot{q}_j$ (A FORMAL DIGRESSION)

LAGRANGE'S EQUATIONS CLEARLY INVOLVE THE $\ddot{q}_j$ – THEY ARE SECOND ORDER DIFFERENTIAL EQUATIONS. BUT TYPICALLY MORE THAN ONE $\ddot{q}_j$ APPEARS IN EACH EQUATION. WE SKETCH A FORMAL PROCEDURE TO ISOLATE EACH $\ddot{q}_j = f_j(q_k, \dot{q}_k)$

WE CONSIDER ONLY THE CASE THAT THE TRANSFORMATIONS TO CARTESIAN COORDINATES CONTAIN NO EXPLICIT TIME DEPENDENCE.

$$\bar{r}_i = \bar{r}_i(q_j) \Rightarrow \bar{v}_i = \sum_j \frac{\partial \bar{r}_i}{\partial q_j} \dot{q}_j$$

THEN $T = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_j \sum_k a_{jk} \dot{q}_j \dot{q}_k$

WHERE THE a_{jK} DEPEND ONLY ON q_i, NOT t EXPLICITLY. $a_{jk} = a_{kj}$

LAGRANGE SAYS: $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$

$$\frac{\partial T}{\partial \dot{q}_j} = \sum_K a_{jK} \dot{q}_K \quad ; \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = \sum_K a_{jK} \ddot{q}_K + \sum_K \sum_l \frac{\partial a_{jk}}{\partial q_l} \dot{q}_K \dot{q}_l$$

NOT 1/2 !

$$\frac{\partial T}{\partial q_j} = \frac{1}{2} \sum_K \sum_l \frac{\partial a_{Kl}}{\partial q_j} \dot{q}_K \dot{q}_l$$

SO $$\sum_K a_{jK} \ddot{q}_K + \frac{1}{2} \sum_K \sum_l \left(\frac{\partial a_{jk}}{\partial q_l} + \frac{\partial a_{jl}}{\partial q_K} - \frac{\partial a_{Kl}}{\partial q_j} \right) \dot{q}_K \dot{q}_l = Q_j$$

THE MAIN AMUSEMENT OF THIS DIGRESSION IS TO NOTE THE APPEARANCE OF THE CHRISTOFFEL SYMBOLS

$$\begin{bmatrix} K & l \\ j & \end{bmatrix} \equiv \frac{1}{2}\left(\frac{\partial a_{jk}}{\partial q_l} + \frac{\partial a_{jl}}{\partial q_K} - \frac{\partial a_{Kl}}{\partial q_j} \right)$$

SO $$\sum_K a_{jK} \ddot{q}_K = Q_j - \sum_K \sum_l \begin{bmatrix} K & l \\ j & \end{bmatrix} \dot{q}_K \dot{q}_l$$

THIS IS A MATRIX EQUATION

IF a^{-1}_{jk} IS THE INVERSE MATRIX, THEN

$$\sum_j a^{-1}_{mj}\, a_{jk} = \delta_{mk} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix}$$

THUS

$$\ddot{q}_m = \sum_j a^{-1}_{mj} Q_j - \sum_j \sum_k \sum_l a^{-1}_{mj} \begin{bmatrix} k & l \\ & j \end{bmatrix} \dot{q}_k \dot{q}_l$$

IN MECHANICS THIS FORMALISM IS OVERKILL, BUT THE CHRISTOFFEL SYMBOLS WILL REAPPEAR IN A PROMINENT WAY IN THE THEORY OF GENERAL RELATIVITY.