

# To Construct a Square with Edges on Any Four Points

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(February 11, 2001; updated January 31, 2014)

## 1 Problem

Given a set of four points on a plane, construct a square with one of the four points on each of its four edges, as shown in Fig. 1.

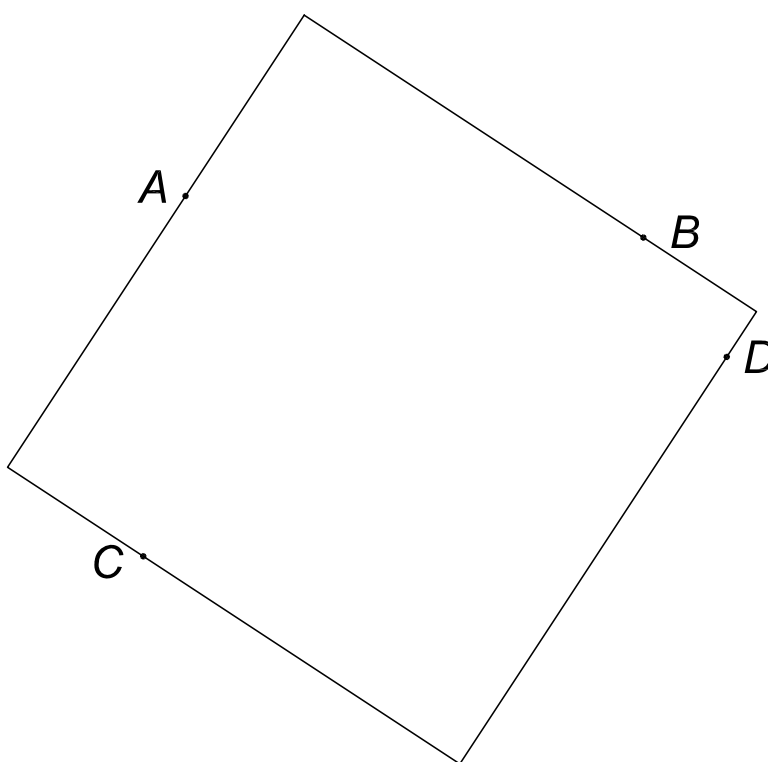


Figure 1: Construct a square with a given set of four points  $A$ ,  $B$ ,  $C$  and  $D$  on its edges.

## 2 Solution

This problem, and its solution, was suggested to us by Konstantin Shmakov.

In the figures, points are labeled by upper case letters, and angles are labeled by lower case letters.

We first prove a geometric theorem, and note two corollaries.

## 2.1 Theorem 0

The angle  $a = \angle APB$  subtended by a chord  $AB$  of a circle with respect to any point  $P$  on the circumference on the same side of the chord as the center of the circle is one half the angle  $b = \angle AOB$  subtended by the chord with respect to the center of the circle; if point  $P$  is on the other side of the chord from the center of the circle then  $a = \pi - b/2$ .

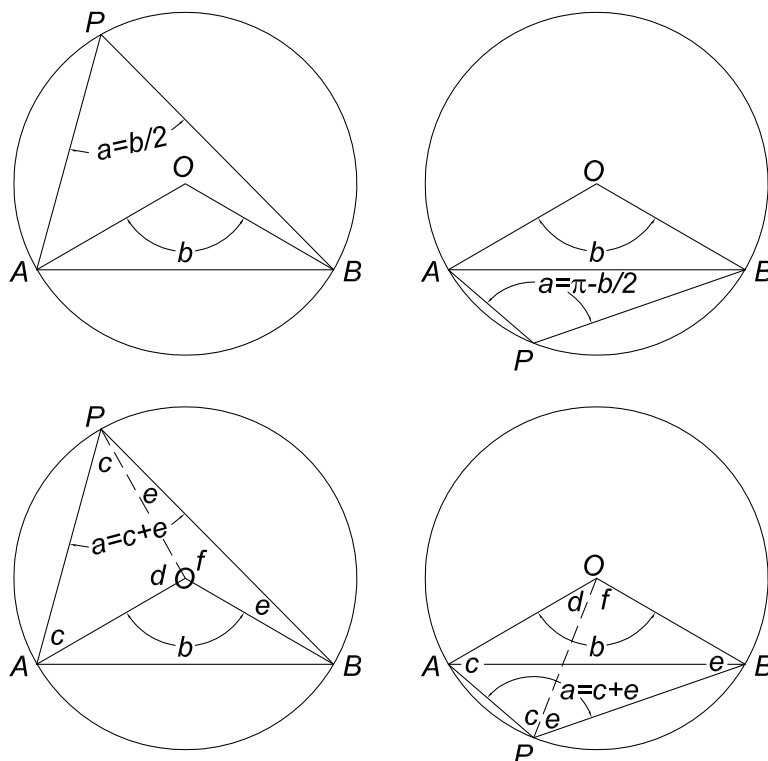


Figure 2: To show that angle  $a = b/2$  (left) or  $\pi - b/2$  (right).

*Proof:* Draw line  $OP$  from the center of the circle to point  $P$ , as shown in the lower portions of Fig. 2. Then,  $\triangle AOP$  and  $\triangle POB$  (and also  $\triangle AOB$ ) are isosceles triangles, such that angle  $c = (\pi - d)/2$ , angle  $e = (\pi - f)/2$ , and  $a = c + e = \pi - (d + f)/2$ . If points  $O$  and  $P$  are on the same side of the chord  $AB$  (left figures above), then  $d + f = 2\pi - b$  and  $a = b/2$ ; while if points  $O$  and  $P$  are on opposite sides of the chord  $AB$  (right figures above), then  $d + f = b$  and  $a = \pi - b/2$ . *QED.*

## 2.2 Theorem 1

The diameter of a circle subtends a  $90^\circ$  angle from any point  $P$  on the circumference.

In Fig. 3, the diameter  $AB$  subtends angle  $b = \pi$  with respect to the center of the circle, so angle  $a$  is  $\pi/2 = 90^\circ$  according to Theorem 0.<sup>1</sup>

<sup>1</sup>Thanks to Christopher Crawford for noting that Theorems 1 and 2 follow from Theorem 0.

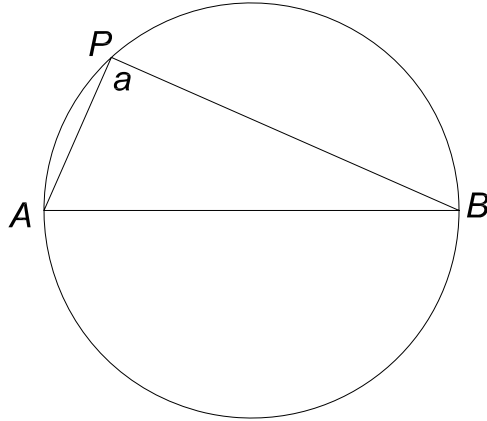


Figure 3: Angle  $a = 90^\circ$  when chord  $AB$  is a diameter.

### 2.3 Theorem 2

The angle  $a$  subtended from any point  $P$  on the circumference of a circle to the endpoints of two orthogonal diameters is  $45^\circ$ , if the point does not lie along the  $90^\circ$  arc between the two endpoints of the diameters. If the point does lie along that arc, the angle subtended is  $135^\circ$ .

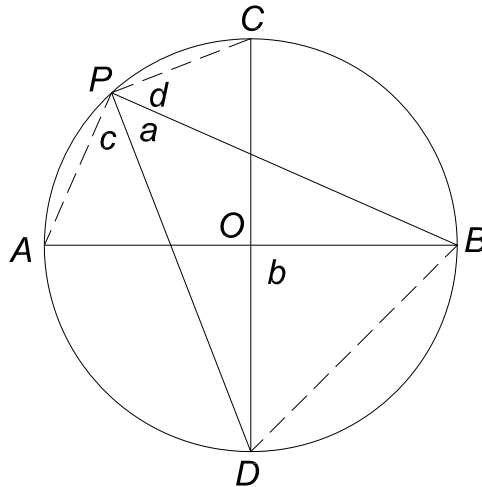


Figure 4: Angles  $a = c = d = 45^\circ$  when chords  $AB$  and  $CD$  are two orthogonal diameters of a circle.

In Fig. 4, the point is  $P$ , and the orthogonal diameters are  $AB$  and  $CD$ . Then, chord  $BD$  subtends angle  $b = 90^\circ$  with respect to the center  $O$  of the circle, and angle  $a = b/2 = 45^\circ$  if  $O$  and  $P$  are on the same side of the chord  $BD$ , according to Theorem 0.

Similarly, chords  $AD$  and  $BD$  subtends angle  $90^\circ$  with respect to the center  $O$  of the circle, and angle  $c = d = 45^\circ$  if  $O$  and  $P$  are on the same side of the chord  $BD$ ,

## 2.4 The Four Point Construction

### 2.4.1 Rectangle Through Four Points Using Theorem 1

An infinite set of rectangles can be fit through four points using the result of Theorem 1, as shown in Fig. 5.

Construct four circles which have diameters as the four sides of the quadrilateral  $ABCD$ . Pick an arbitrary point  $I$  on the arc  $AB$ , and extend a line from point  $I$  through point  $B$  until it intersects arc  $BC$  at point  $J$ . Extend a line from point  $J$  through point  $C$  until it intersects arc  $CD$  at point  $K$ . Then  $\angle BJC = 90^\circ$  according to Theorem 1.

Extend a line from point  $K$  through point  $D$  until it intersects arc  $DA$  at point  $L$ , forming right angle  $CKD$ . Finally, extend a line from point  $L$  through point  $A$  until it intersects arc  $AB$  at point  $I$ , forming right angles  $LDA$  and  $AIB$ . Then, quadrilateral  $IJKL$  is a rectangle with points  $A, B, C,$  and  $D$  on its four sides.

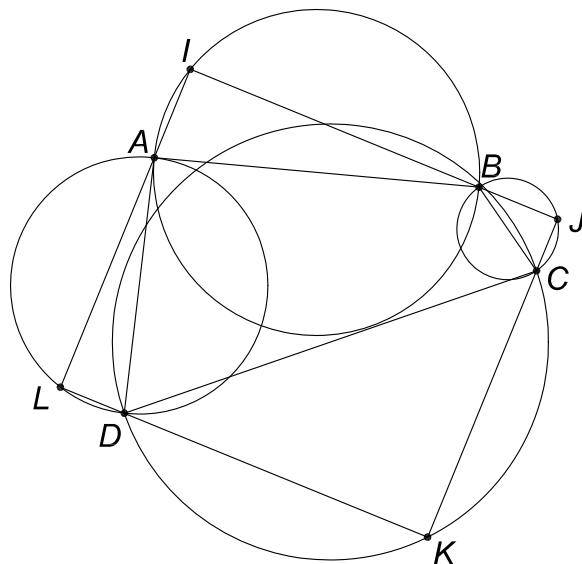


Figure 5: Construction of a rectangle  $IJKL$  whose sides contain the four points  $A, B, C,$  and  $D,$  using Theorem 1.

However, in general the rectangles  $IJKL$  are not squares.

### 2.4.2 Square Through Four Points Using Theorem 2

We seek a method to choose among the infinite set of rectangles  $IJKL$  that contain points  $A, B, C,$  and  $D$  those that are squares.

In general, the diagonal  $IK$ , shown in Fig. 6, of rectangle  $IJKL$  does not bisect the right angles  $AIB$  and  $CDK$ , but if it did the rectangle would be a square. That is, we desire that  $\angle AIK, \angle BIK, \angle CKI,$  and  $\angle DKI$  all be  $45^\circ$ . This suggests use of Theorem 2.

For example, consider the two circles whose diameters are the lines  $AB$  and  $CD$ , as shown in Fig. 7. Construct diameters  $EF$  perpendicular to  $AB$  and  $GH$  perpendicular to

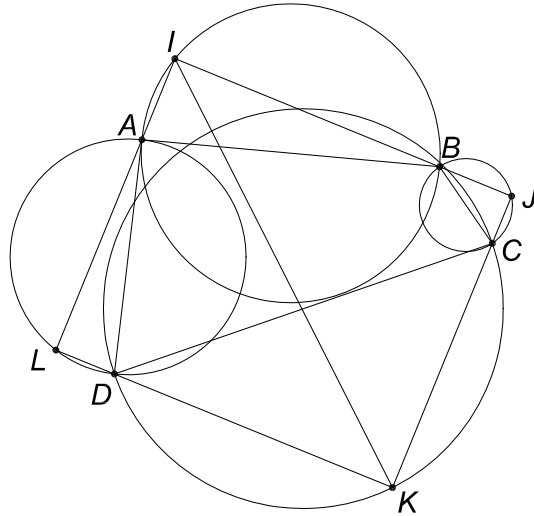


Figure 6: The rectangle  $IJKL$  would be a square if the diagonal  $IK$  bisected the right angles  $AIB$  and  $CKD$ .

$CD$ . Then, extend line  $FH$  until it intersects arc  $AB$  at point  $I$  and arc  $CD$  at point  $K$ . Theorem 2 tells us that  $\angle AIK$ ,  $\angle BIK$ ,  $\angle CKI$ , and  $\angle DKI$  are all  $45^\circ$  as desired.

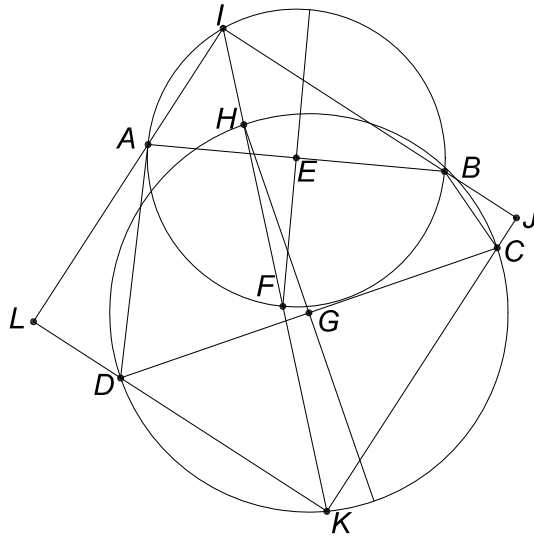


Figure 7: Construction of a square  $IJKL$  whose sides contain the four points  $A$ ,  $B$ ,  $C$ , and  $D$ , using Theorem 2.

We complete the construction of a square by extending lines  $IB$  and  $CD$  to intersect at point  $J$ , and lines  $AI$  and  $DK$  to intersect at point  $L$ . Since  $\angle JIK$  and  $\angle JKI$  of  $\triangle IJK$  are both  $45^\circ$ ,  $\angle IJL = 90^\circ$ . Similarly,  $\angle KLI = 90^\circ$ , and so the quadrilateral  $IJKL$  is a square which contains points  $A$ ,  $B$ ,  $C$ , and  $D$  on its sides. *QED*

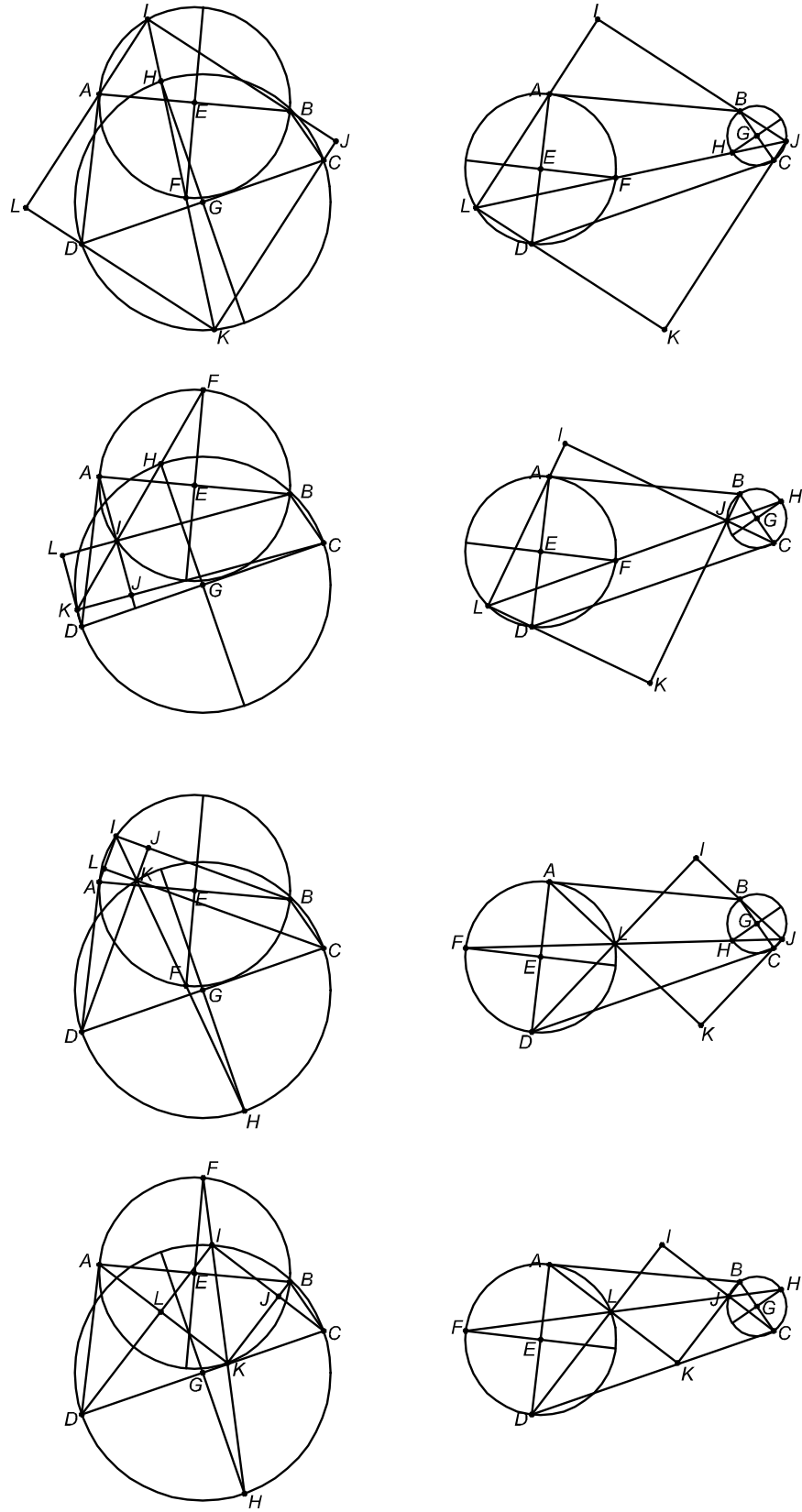


Figure 8: The 8 variants of the construction of a square  $IJKL$  whose (extended) sides contain the four points  $A$ ,  $B$ ,  $C$ , and  $D$ .

