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Bunch-Timing Measurement in the Muon Cooling Experiment Via a Rectangular $TM_{2,1,0}$ RF Deflection Cavity

1 Introduction

In the previous note, Princeton/$\mu\mu$/97-5, we studied the transverse displacement of a muon on passing through a rectangular $TE_{0,1,1}$ RF cavity as a function of longitudinal position within the bunch. However, the effect was rather small. Here we consider a rectangular $TM_{2,1,0}$ cavity in which a muon is deflected by the magnetic field, which is large at the center of the cavity. The angular deflection is about three times that of the effective angular deflection in a $TE_{0,1,1}$ cavity, which permits the tracking system and cavity wall to be about ten times thicker for the same time resolution.

2 Rectangular $TM_{2,1,0}$ Cavity Fields

The rf cavity is centered on $(x, y, z) = (0, 0, 0)$, and is a rectangular box of length a in x (the direction of transverse deflection), and length a/α in y and length b along the beam direction z.

The trajectory of a typical beam particle for the cavity field OFF is parametrized as

$$\begin{aligned} x &= x_0 + \beta_x ct, \\ y &= y_0 + \beta_y ct, \\ z &= z_0 + \beta_z ct, \end{aligned} \tag{1}$$

where c is the speed of light. The beam axis is the z-axis:

$$\beta_x,\ \beta_y \ll \beta_z, \qquad \text{and} \qquad \beta_z \approx \beta. \tag{2}$$

We will make the impulse approximation that the cavity fields do not affect the muon trajectories in y or z, but only in x. Thus we assume the y and z parametrizations in (1) also hold when the field in ON.

The particle is within the cavity during the interval

$$[t_{\min}, t_{\max}] = \left[-\frac{b}{2\beta_z c} - \frac{z_0}{\beta_z c}, \frac{b}{2\beta_z c} - \frac{z_0}{\beta_z c}\right]. \tag{3}$$

The wave equation tells us that for a $\mathrm{TM}_{2,1,0}$ cavity

$$\frac{\omega}{c} = \sqrt{4+\alpha^2}\frac{\pi}{a}, \qquad \text{so} \qquad a = \frac{\sqrt{4+\alpha^2}}{2}\lambda, \tag{4}$$

where $\omega =$ is the angular frequency and $\lambda = c/2\pi\omega$ is the wavelength. For frequency $\nu = 800$ MHz, $\lambda = 37.5$ cm.

The cavity is phased so that the magnetic field is minimum at $t = 0$. In Gaussian units,

$$\begin{aligned}
E_x = E_y &= 0, \\
E_z &= E_0 \sin\frac{2\pi x}{a}\cos\frac{\pi\alpha y}{a}\cos\omega t, \\
B_x &= \frac{\alpha E_0}{\sqrt{4+\alpha^2}}\sin\frac{2\pi x}{a}\sin\frac{\pi\alpha y}{a}\sin\omega t, \\
B_y &= \frac{2E_0}{\sqrt{4+\alpha^2}}\cos\frac{2\pi x}{a}\cos\frac{\pi\alpha y}{a}\sin\omega t, \\
B_z &= 0.
\end{aligned} \tag{5}$$

3 Transverse Deflection: Leading Approximation

To a good approximation the energy of the muon does not change in the RF cavity; $\gamma = 1/\sqrt{1-\beta^2}$ remains constant. The x-component of the Lorentz-force law can then be written

$$\begin{aligned}
\frac{d\beta_x}{dt} &= \frac{e}{\gamma mc}(E_x + \beta_y B_z - \beta_z B_y) = -\frac{\beta_z e B_y}{\gamma mc} \\
&= -\frac{2\beta_z e E_0}{\gamma\sqrt{4+\alpha^2}mc}\cos\frac{2\pi x}{a}\cos\frac{\pi\alpha y}{a}\sin\omega t \approx -\frac{2\beta_z e E_0}{\gamma\sqrt{4+\alpha^2}mc}\sin\omega t,
\end{aligned} \tag{6}$$

using eq. (5) and supposing that the transverse size of the beam is small compare to a.

From eq. (6) we see that the angular kick is x depends on the x and y positions and slopes only in second order.

The change in the x-velocity due to the RF cavity is

$$\begin{aligned}
\Delta\beta_x &= \int_{t_{\min}}^{t_{\max}} \frac{d\beta_x}{dt}dt \approx -\frac{2\beta_z e E_0}{\gamma\sqrt{4+\alpha^2}mc}\int_{t_{\min}}^{t_{\max}} \sin\omega t\ dt \\
&= \frac{2\beta_z}{\gamma\sqrt{4+\alpha^2}}\frac{eE_0}{m\omega c}(\cos\omega t_{\max} - \cos\omega t_{\min}) \approx \frac{8\pi}{\gamma\sqrt{4+\alpha^2}}\frac{\eta z_0}{\lambda}\sin\frac{\omega b}{2\beta_z c},
\end{aligned} \tag{7}$$

using eq. (3) and introducing the dimensionless measure of field strength

$$\eta = \frac{eE_0}{m\omega c} = \frac{eE_0}{mc^2}\frac{c}{2\pi\nu}. \tag{8}$$

We choose length b so that the argument of the sine is $\pi/2$:

$$b = \frac{\pi\beta_c c}{\omega} = \frac{\beta_z\lambda}{2} = 15.75 \text{ cm} \tag{9}$$

for $\nu = 800$ MHz and $\beta_z = 0.84$, corresponding to muon momentum of 165 MeV/c. Thus a muon is in the cavity for half a cycle. Then,

$$\Delta\beta_x \approx \frac{8\pi}{\gamma\sqrt{4+\alpha^2}}\frac{\eta z_0}{\lambda}. \tag{10}$$

The corresponding angular deflection is

$$\Delta\theta_x = \frac{\Delta\beta_x}{\beta_z} \approx \frac{8\pi}{\gamma\beta_z\sqrt{4+\alpha^2}}\frac{\eta z_0}{\lambda} = \frac{8\pi}{\gamma\sqrt{4+\alpha^2}}\frac{\eta c\Delta t}{\lambda}, \tag{11}$$

introducing the time offset $\Delta t = z_0/\beta_z c$ of the muon from the center of the bunch.

4 Discussion

To maintain x-y symmetry we consider aspect ratio $\alpha = 2$, for which $a = 53$ cm at 800 MHz, and

$$\Delta\theta_x \approx \frac{2\sqrt{2}\pi}{\gamma}\frac{\eta c\Delta t}{\lambda}. \tag{12}$$

As an example we consider a peak field $E_0 = 40$ MV/m, corresponding to $\eta = 0.0223$, and 165-MeV/c muons for which $\gamma = 1.85$. Then

$$\Delta\theta_x \approx \frac{2\sqrt{2}\pi \cdot 0.0223 \cdot 3\times 10^{-4}\text{ m/ps}}{1.85\cdot 0.375\text{ m}}\Delta t[\text{ps}] = 86\ \mu\text{rad}\left[\frac{\Delta t}{1\text{ ps}}\right]. \tag{13}$$

This is nearly three times the angular deflection found for a $\text{TE}_{0,1,1}$ cavity (eq. (63) in our note Princeton/$\mu\mu$-97-5). If we used $\alpha = 1$ the angular deflection would be 26% larger.

It will be difficult the measure the small angular deflections the arise from a single cavity. However, if we make a multicell RF deflection structure (with irises to let the beam pass) the angular deflection would be cumulative, to our advantage.

Continuing the analysis of a single cavity, we suppose the timing resolution function is well understood and so we relax the requirement on timing resolution to, say $\sigma_{t,D} = 0.5\sigma_t = 20$ ps, following Table 1 of our note Princeton/$\mu\mu$/97-4. This requires knowing the timing resolution function to 4%. Then the corresponding angular resolution is $\sigma_{\theta_x,D} = 1.7$ mrad. Then the material in the cavity wall and tracking system just upstream must satisfy

$$X_0 < \left(\frac{\sigma_{\theta_x,D}P\beta_z}{15\text{ MeV}/c}\right)^2 = \left(\frac{0.0017\cdot 165\cdot 0.84}{15}\right)^2 = 0.00025 \text{ radiation lengths.} \tag{14}$$

The cavity wall could then be about 75 μm thick if made of beryllium.