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POLARIZATION PRECESSION *

by

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POLARIZATION PRECESSION

Many high energy experiments measure the spin polarization or other quantities depending on the polarization of a particle. If the particle passes through an electromagnetic field (e.g. a magnet spectrometer) the spin may precess. We derive an expression for the precession for arbitrary fields, polarizations and particle velocities. Previous derivations of the precession[1-5] are restricted to velocities either normal or parallel to the magnetic field.

Rather than use the Dirac equation[2,3,4] (or some relativistic wave equation for higher spin) to calculate the precession, we first consider the (instantaneous) rest frame of the particle, where non-relativistic wave equations apply.[1,5] The results may then be transformed to the lab with complete generality. However, if we construct a polarization four-vector and transform it to the lab frame, the meaning of the various lab components of the four-vector is unclear. One usually returns to the rest frame to understand the polarization vector. Our approach is to always deal with the physically meaningful rest frame polarization vector, but to use the laboratory values of all other quantities.

Let us illustrate that the rest frame polarization is indeed the physically meaningful quantity to a lab observer. Consider the decay of a polarized particle. Calculations of the correlation between the polarization and the decay angular distribution are always made in the rest frame of particle. As another example consider the two body scattering involving a spin 1/2 particle whose polarization is to be measured. The only parity conserving operator in the scattering amplitude involving the spin is $\vec{\sigma} \cdot \hat{n}$, where $\hat{n}$ is the normal to the scattering plane. Thus only the component of the spin transverse to the particle's direction is significant. This component is the same in the rest, lab and center of mass frames, so that calculating spin direction in the particle's rest frame is valid.

In the rest frame of the particle, then, the polarization, $\vec{P}$, obeys

$$\frac{d\vec{P}}{d\tau} = g \frac{e}{2m} \vec{P} \times \vec{B}^* \tag{1}$$

where τ = proper time

g = gyromagnetic ratio ($g = 2(2.79)$ for protons)

e = charge

m = mass

$\vec{B}^*$ = magnetic field in the rest frame

This equation follows from classical electromagnetism using the correspondence principle.

We illustrate this for spin-1/2 particles. We take $\vec{P} = \langle\vec{\sigma}\rangle$
Now, $\frac{d\langle\vec{\sigma}\rangle}{d} = \left\langle\frac{d\vec{\sigma}}{dt}\right\rangle$ and $\frac{d\vec{\sigma}}{dt} = \frac{i}{\hbar}\left[H,\vec{\sigma}\right]$ where H is the Hamiltonian. Since we are in the rest frame, we may use the non-relativistic Hamiltonian. The only part of the Hamiltonian for a free particle in an electromagnetic field which does not commute with $\vec{\sigma}$ is $H' = -\frac{g}{2}\frac{e\hbar}{2m}\vec{\sigma}\cdot\vec{B}$

$$\frac{d\vec{\sigma}}{dt} = -i\,\frac{g}{2}\frac{e}{2m}\left[\vec{\sigma}\cdot\vec{B},\ \vec{\sigma}\right]$$

$$= g\,\frac{e}{2m}\,\vec{\sigma} \times \vec{B}.$$

Hence equation (1).

The next step is to convert τ and $\vec{B}^*$ to their corresponding values as seen in the lab frame. We do not convert $\vec{P}$ to the lab frame as mentioned above. Let the instantaneous 3-velocity of the particle be $\vec{\beta}$ with respect to the lab frame and let $\beta = |\vec{\beta}|$; $\gamma = \frac{1}{\sqrt{1-\beta^2}}$; and $\hat{\ell} = \vec{\beta}/\beta$ = unit tangent to the particle's trajectory. The Lorentz transformation from the rest frame to the frame gives

$$d\tau = \frac{1}{\gamma}\, dt \tag{2}$$

where t is the time in the lab frame. Also

$$\vec{B}^* = \vec{B}_{11} + \gamma(\vec{B}_{\perp} - \vec{\beta} \times \vec{E})$$

where $\vec{B}_{11} = (\vec{B} \cdot \hat{\ell})\hat{\ell}$ is the longitudinal part of the lab magnetic field;

$\vec{B}_{\perp} = \vec{B} - (\vec{B} \cdot \hat{\ell})\hat{\ell}$ is the transverse part: and

$\vec{E}$ is the lab electric field. Thus

$$\vec{B}^* = (\vec{B} \cdot \hat{\ell})\hat{\ell} + \gamma(\vec{B} - (\vec{B} \cdot \hat{\ell})\hat{\ell} - \beta\hat{\ell} \times \vec{E}) \tag{3}$$

Substituting (2) and (3) into (1), the precession in the rest frame is

$$\frac{d\vec{P}}{dt} = g\,\frac{e}{2m}\,\vec{P} \times \left[\vec{B} - (\vec{B} \cdot \hat{\ell})\hat{\ell} - \beta\hat{\ell} \times \vec{E} + \frac{1}{\gamma}(\vec{B} \cdot \hat{\ell})\hat{\ell}\right] \tag{4}$$

At this point, we should distinguish between two different "rest" frames. The first, to be called the <u>rest</u> frame, is that frame in which the particle is at rest and in which in the absence of external torques the spin would not precess. The second frame, to be called the <u>comoving</u> frame, can be thought of as a succession of inertial frames each with velocity equal to the instantaneous particle velocity and orientation tangent to the particle's direction. Of course the particle is at rest in the comoving frame, so that the comoving and rest frame can differ at most by a rotation. We began our considerations in the rest frame and now switch to the comoving frame which is more closely related to the lab trajectory of the particle.

In the Appendix it is shown that a vector, $\vec{s}$, which is constant in the rest frame obeys

$$\frac{d\vec{s}}{dt} = \gamma\, \vec{s} \times \vec{\omega} \tag{5}$$

relative to the comoving frame, where

$$\vec{\omega} = \hat{\ell} \times \frac{d\hat{\ell}}{dt} \tag{6}$$

That is, $\vec{s}$ precesses with frequency $\gamma\omega$ relative to the coordinate triad formed by $\hat{\ell}$, $\frac{d\hat{\ell}}{dt}$ and $\vec{\omega}$ in the comoving frame. This is the so-called Thomas precession. Equation (4) in the comoving frame is thus

$$\left.\frac{d\vec{P}}{dt}\right|_{\text{comoving}} = \left.\frac{d\vec{P}}{dt}\right|_{\text{rest}} + \gamma\vec{P} \times \vec{\omega} \tag{7}$$

To calculate $\vec{\omega}$ from equation (6), we need $\frac{d\hat{\ell}}{dt}$. For this, we use the relativistic form of the Lorentz force: $\frac{du}{dt} = -\frac{e}{m} Fu$ where u is the four-velocity: $u = (\gamma, \gamma\vec{\beta})$, and F is the electromagnetic field tensor:

$$F = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

The resulting equations are

$$\frac{d\gamma}{dt} = \frac{e}{m}\,\beta\,\vec{E}\cdot\hat{\ell}$$

$$\frac{d(\gamma\beta\hat{\ell})}{dt} = \frac{e}{m}\,(\vec{E} + \beta\hat{\ell} \times \vec{B}) \quad \text{using } \vec{\beta} = \beta\hat{\ell}$$

Solving for $d\hat{\ell}/dt$ we get

$$\frac{d\hat{\ell}}{dt} = \frac{e}{m}\left[\frac{\vec{E} - (\vec{E}\cdot\hat{\ell})\hat{\ell}}{\gamma\beta} + \frac{\hat{\ell} \times B}{\gamma}\right] \tag{8}$$

and, from equation (6),

$$\vec{\omega} = -\frac{e}{m}\left[\frac{\vec{B} - (\vec{B}\cdot\hat{\ell})\hat{\ell}}{\gamma} + \frac{\hat{\ell} \times \vec{E}}{\gamma\beta}\right] \tag{9}$$

Expression (8) is strictly true only in homogeneous fields.[6] In an inhomogeneous magnetic field there is an additional force on the particle due to the magnetic moment interaction, $\vec{\nabla}(\vec{\mu}\cdot\vec{B})$. However, this is extremely small compared to the usual magnetic force: $ec\vec{\beta} \times \vec{B}$. For example, suppose $\vec{\mu}$ is parallel to $\vec{B}$, and $\vec{\beta}$ is normal to $\vec{B}$. Then the ratio of these two terms is $\frac{\mu}{\beta ec}\frac{\nabla B}{B} \simeq \frac{\hbar}{2mc}\frac{1}{\beta}\frac{\nabla B}{B} \simeq \frac{\lambda}{\beta}\frac{\nabla B}{B}$

using $\mu = e\hbar/2m$, and $\lambda = \hbar/mc$, the Compton wavelength. $\nabla B/B$ might be as great as 0.1 for $B = 10$ KG and $\nabla B = 1$ KG in 1 cm. For protons $\lambda \simeq 10^{-13}$ cm. So, for $\beta = 0.1$ protons (45MeV) the ratio is of the order of 10^{-13}. Hence the formulas derived in this note are applicable to ordinary non-homogeneous fields.

Substituting equation (9) into (7) gives

$$\frac{d\vec{P}}{dt} = \frac{e}{m}\left(\frac{g}{2} - 1\right)\vec{P} \times (\vec{B} - (\vec{B}\cdot\hat{\ell})\hat{\ell}) + \frac{e}{m\gamma}\frac{g}{2}\vec{P} \times (\vec{B}\cdot\hat{\ell})\hat{\ell}$$

$$- \frac{e}{m\gamma\beta}\left[\gamma\left(\frac{g}{2} - 1\right) - \frac{g}{2\gamma}\right]\vec{P} \times (\hat{\ell} \times \vec{E}) \qquad (10)$$

This is the main result for precession of the polarization with respect to the comoving frame. If $d\vec{P}/dt = 0$ in equation (10), $\vec{P}$ is constant with repsect to the coordinate triad formed by $\hat{\ell}$, $d\hat{\ell}/dt$ and $\vec{\omega}$.

From now on, we put $\vec{E} = 0$.

Suppose $\hat{\ell}$ is parallel to $\vec{B}$. Then $\frac{d\vec{P}}{dt} = g\,\frac{e}{2m\gamma}\,\vec{P} \times \vec{B}$ which is just the classical result except for the time dilation factor γ. This effect has been utilized in measurements of the electron's gyromagnetic ratio.[7]

Suppose $\hat{\ell}$ is perpendicular to $\vec{B}$. Then $\frac{d\vec{P}}{dt} = \frac{e}{m}\left(\frac{g}{2} - 1\right)\vec{P} \times \vec{B}$. A Dirac particle would not precess at all in this case. From equation (9) $\frac{d\hat{\ell}}{dt} = \frac{e}{m\gamma}\hat{\ell} \times B$. Thus the precession of $\vec{P}$ is $\gamma\left(\frac{g}{2} - 1\right)$ times that of $\hat{\ell}$. Thus if the particle is bent by Θ_ℓ, the polarization precesses by

$$\Theta_p = \gamma\left(\frac{g}{2} - 1\right)\Theta_\ell \qquad (11)$$

This result is useful for quick calculations as to the effect of a bending magnet on the polarization. For a slow proton, (11) gives $\Theta_p \simeq 2\,\Theta_\ell$. Hence a 45^0 bending magnet can interchange transverse and longitudinal polarizations if $\vec{P}$ is perpendicular to $\vec{B}$, while a 90^0 magnet merely changes the sign of the polarization.[8] Of course if $\vec{P}$ is parallel to $\vec{B}$, the precession has no physical effect. The frequency of precession is

$\frac{e}{m} \cdot (\frac{g}{2} - 1)B$, which does not depend on γ. Hence in a magnetic field a fast muon would undergo more spin precession than a slow muon in its lifetime, which is proportional to γ. This allows a more precise determination of the μ^- gyromagnetic ratio.[9]

We now consider the case where $\hat{\ell}$ is neither perpendicular nor parallel to $\vec{B}$, so that the trajectory is helical. From equation (10) we see that the polarization no longer precesses about $\vec{B}$, but rather about $\vec{B}' = \vec{B}_\perp + \frac{g}{2\gamma(\frac{g}{2} - 1)} \vec{B}_{11}$, a vector in the plane formed by $\vec{B}$ and $\hat{\ell}$. If we designate the angle between $\vec{B}$ and $\hat{\ell}$ by Θ, and the angle between $\vec{B}'$ and $\hat{\ell}$ by Θ', then

$$\tan \Theta' = (\frac{g}{2} - 1)\frac{2\gamma}{g} \tan \Theta$$

$$= .64\ \gamma \tan \Theta \text{ for protons.}$$

When $\gamma = 1.56$, $\tan \Theta' = \tan \Theta$ and $\vec{B}' = \vec{B}$. For protons, $\gamma = 1.56$ occurs at a kinetic energy of 525 MeV. As $\gamma \to \infty$, $\tan \Theta' \to 1$ and $\vec{P}$ precesses about the transverse part of $\vec{B}$. For applications to many bending magnets, $\hat{\ell}$ will be nearly normal to $\vec{B}$ and this effect will be small. A first order correction to equation (11) is possible by making an "effective edge" approximation to the magnetic field. That is, replace the actual, spatially-varying field with a constant field which produces the same total bend in the particle trajectory. Then $\vec{B} \cdot \hat{\ell}$ is a constant, so that $\vec{B}'$ is constant in magnitude, and precesses about $\vec{B}$ at the same frequency as $\hat{\ell}$.

The corrected form of equation (11) is

$$\Theta_p = \gamma(\frac{g}{2} - 1)\ \Theta_p\ |\vec{B}'|/|\vec{B}|$$

or

$$\Theta_p = \gamma\left(\frac{g}{2} - 1\right) \Theta_\ell \sqrt{1 + \cos^2 \Theta \left(\frac{g}{\gamma(g - 2)} - 1\right)^2} \qquad (12)$$

where Θ is the angle between $\vec{B}$ and $\hat{\ell}$. Note, however, that in equation (11), both Θ_p and Θ_ℓ are azimuthal angles measured with respect to $\vec{B}$, while in (12), Θ_p is with respect to $\vec{B}'$.

APPENDIX: THE THOMAS PRECESSION

We wish to find the time dependence in the comoving frame of a vector which is constant in the particle's rest frame.

Consider the four-vector $s = (0,\vec{s})$ in the rest frame, satisfying $\frac{d\vec{s}}{d\tau} = 0$. Let u be the four-velocity of the particle, so that in the rest frame, $u = (1,\vec{0})$. Thus in the rest frame $u \cdot s = 0$, so this is true in all frames. Further, $\frac{d(u \cdot s)}{d\tau} = 0$, or $\frac{ds}{d\tau} \cdot u = -s \cdot \frac{du}{d\tau}$

This suggests

$$\frac{ds}{d\tau} = -(s \cdot \frac{du}{d\tau})\ u \tag{A1}$$

as the covariant form of the rest frame condition $\frac{d\vec{s}}{d\tau} = 0$. By considering this equation in the rest frame, we see that it is correct.

Consider the lab frame where $s = (s_L^o, \vec{s}_L)$ and $u = (\gamma,\gamma\vec{\beta})$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

The condition $u \cdot s = 0$ implies $s_L^o = \vec{s}_L \cdot \vec{\beta}$.
Defining $\hat{\ell} = \vec{\beta}/\beta$ we can write, using (A1),

$$\begin{aligned}\frac{d\vec{s}_L}{d\tau} &= -\left(s_L^o \frac{d\gamma}{d\tau} - \vec{s}_L \cdot \frac{d\gamma\beta\hat{\ell}}{d\tau}\right)\gamma\beta\hat{\ell} \\ &= (\gamma^2 - 1)\left(\vec{s}_L \cdot \frac{d\hat{\ell}}{d\tau}\right)\hat{\ell} + \frac{1}{\gamma}\frac{d\gamma}{d\tau}\left(\vec{s}_L \cdot \hat{\ell}\right)\hat{\ell}\end{aligned} \tag{A2}$$

Thus in the lab frame, $\vec{s}_L$ does not appear to be precessing; a lab observer does not see the Thomas precession, although $d\vec{s}_L/d\tau \neq 0$.

Consider, however, the comoving frame, which is instantaneously related to the lab frame by a Lorentz transformation in the direction $\hat{\ell}$.

In the lab frame $s = (\vec{s}_L \cdot \vec{\beta}, \vec{s}_L)$

In the comoving frame $s = (0, \vec{s}_c)$

The Lorentz transformation gives

$$\vec{s}_L = \vec{s}_c + (\gamma - 1)(\vec{s}_c \cdot \hat{\ell})\hat{\ell} \tag{A3}$$

$$\vec{s}_c = \vec{s}_L + (\frac{1}{\gamma} - 1)\ (\vec{s}_L \cdot \hat{\ell})\hat{\ell} \tag{A4}$$

Differentiating (A4) with respect to τ and using (A2) gives

$$\frac{d\vec{s}_c}{d\tau} = (\gamma - 1)(\vec{s}_L \cdot \frac{d\hat{\ell}}{d\tau})\hat{\ell} + (\frac{1}{\gamma} - 1)(\vec{s}_L \cdot \hat{\ell})\ \frac{d\hat{\ell}}{d\tau}$$

Equation (A3) then can be used to find

$$\frac{d\vec{s}_c}{d\tau} = (\gamma - 1)\ \left[(\vec{s}_c \cdot \frac{d\hat{\ell}}{d\tau})\hat{\ell} - (\vec{s}_c \cdot \hat{\ell})\frac{d\hat{\ell}}{d\tau}\right]$$

or

$$\frac{d\vec{s}_c}{d\tau} = (\gamma - 1)\ \vec{s}_c \times (\hat{\ell} \times \frac{d\hat{\ell}}{d\tau}) \tag{A5}$$

Thus $\vec{s}_c$ precesses uniformly according to a comoving observer. This is the famous Thomas precession.

In the lab frame, $\frac{d\hat{\ell}}{d\tau} = (\hat{\ell} \times \frac{d\hat{\ell}}{d\tau}) \times \hat{\ell}$ so that $\hat{\ell} \times \frac{d\hat{\ell}}{d\tau}$ has the significance of being the instantaneous rotation of the trajectory. Since $\frac{d\hat{\ell}}{d\tau}$ is transverse to $\hat{\ell}$, this expression holds in the comoving frame also. Thus the part of $\frac{d\vec{s}_c}{d\tau}$ coming from the -1 in the factor $\gamma - 1$ of equation (A5) is due to the rotation of the trajectory in the comoving frame.

Hence the precession of $\vec{s}_c$ relative to the precession of the trajectory is

$$\frac{d\vec{s}_c}{d\tau} = \gamma\, \vec{s}_c \times (\hat{\ell} \times \frac{d\hat{\ell}}{d\tau}) \qquad (A6)$$

in the comoving frame.

The covariant description of the precession used in the appendix may be extended to include the spin precession caused by magnetic fields. This is the approach of Bargmann, Michel and Teledgi[5] which is expanded upon by Hagedorn.[10] The specialization to the comoving frame is straightforward, but lengthy. However, they do not do it. The result of such analysis is equation (10).

J. KELLER ~ 58
ANNALS OF PHYSICS

REFERENCES

1. H. A. Kramers, *Quantum Mechanics*. (North Holland Publishing Company, Amsterdam, 1957) p. 226 ff

2. H. A. Tolhoek and S. R. De Groot, Physica 17, 17, (1951); H. A. Tolhoek, Rev. of Mod. Phys. 28, 277 (1956)

3. H. Mendlowitz and K. M. Case, Phys. Rev. 97, 33 (1955)

4. M. Carrassi, Nuovo Cimento 7, 524 (1958)

5. V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Letters 2, 435 (1959)

6. R. H. Good, Jr., Phys. Rev. 125, 2112 (1962)

7. W. H. Louisell, R. W. Pidd, and H. R. Crane, Phys. Rev. 94, 7 (1954);
   A. A. Schupp, R. W. Pidd and H. R. Crane, Phys. Rev. 121, 1 (1961)

8. D. E. Lundquist, R. L. Anderson, J. V. Allaby, and D. M. Ritson, Phys. Rev. 168, 1527 (1968)

9. F. J. M. Farley, J. Bailey, R. C. A. Brown, M. Giesch, J. Jostlein, S. van der Meer, E. Picasso and M. Tannebaum, Nuovo Cimento 45, 281 (1966)

10. R. Hagedorn, *Relativistic Kinematics*. (W. A. Benjamin, Inc., New York, 1964), chap. 9.