

# Voltage Across the Terminals of a Receiving Antenna

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## 1 Problem

Deduce the no-load (open-circuit) voltage  $V_{oc}$  across the terminals of a short, center-fed linear dipole antenna of half height  $h$  when excited by a plane wave of wavelength  $\lambda \gg h$  whose electric field vector  $\mathbf{E}_{in}$  is parallel to the dipole antenna.

The ratio  $H_{eff} = |V_{oc}/E_{in}|$  is called the **effective height** of the antenna.<sup>1</sup>

Also deduce the current  $I_{sc}$  that would flow between the terminals if they were short circuited. Then, according to Thévenin's theorem the receiving antenna acts on any load connected to it like a voltage source  $V_{oc}$  with internal impedance  $Z_A = V_{oc}/I_{sc}$ .

You may assume that the antenna conductors have a diameter small compared to the height  $h$ , and that they are perfect conductors. The gap between the terminals is also small compared to  $h$ .

*By dimensional analysis, the no-load voltage has amplitude of order  $E_0 h$ , where  $E_0$  is the amplitude of the incident wave. The problem is to show that to a good approximation the voltage is actually  $E_0 h$ . This problem can be addressed using techniques that are simplifications of those appropriate for antennas comprised of thick wires with complex geometries.*

## 2 Solution

The spirit of the solution is due to Pocklington [4], who extended the insights of Lorenz [5] and Hertz [6] that electromagnetic fields can be deduced from the retarded vector potential, by consideration of the boundary condition that the tangential component of the electric field must vanish at the surface of a good/perfect conductor. Furthermore, Pocklington noted that to a good first approximation for conductors that are thin wires, the vector potential at the surface of a wire depends only on the current in the wire at that point. Pocklington deduced an integral equation for the currents in the conductors, which equation has been elaborated upon by L.V. King [7], E. Hallén [8] and R.W.P. King [9, 10, 11, 12] to become the basis of numerical electromagnetic codes such as NEC4 [13]. See also [14], on which this solution is based.

In the present example an incident electromagnetic wave with electric field

$$\mathbf{E}_{in} = E_0 e^{-i(kx - \omega t)} \hat{\mathbf{z}} \quad (1)$$

excites an oscillating current distribution  $\mathbf{J}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r})e^{i\omega t}$  in the conductors of the receiving antenna. If this current distribution is known, then the retarded vector potential  $\mathbf{A}(\mathbf{r}, t) =$

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<sup>1</sup>Schelkunoff [1, 2] has defined an effective length vector  $\mathbf{H}_{eff}$  for transmitting antennas in terms of their far-zone electric field  $\mathbf{E}_0(\theta, \phi)e^{-i(kr - \omega t)}/r$  as  $\mathbf{H}_{eff}(\theta, \phi) = ic\mathbf{E}_0/kI_0$ , where the current at the antenna terminals is  $I_0e^{i\omega t}$ , and  $\mathbf{E}_0$  and  $I_0$  are complex quantities in general. For a linear antenna, the magnitude  $H_{eff}(\theta = 90^\circ)$  equals the effective height of eq. (15), as can be confirmed using eqs. (10) and (61) of [3].

$\mathbf{A}(\mathbf{r})e^{i\omega t}$  of the response fields can be calculated as

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t' = t - \mathcal{R}/c)}{\mathcal{R}} d\text{Vol}' = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \frac{e^{ik\mathcal{R}}}{\mathcal{R}} d\text{Vol}' e^{i\omega t} = \mathbf{A}(\mathbf{r})e^{i\omega t}, \quad (2)$$

where  $\mathcal{R} = |\mathbf{r} - \mathbf{r}'|$ ,  $c$  is the speed of light,  $\omega$  is the angular frequency,  $k = \omega/c$  is the wave number, and the medium outside the conductors is vacuum whose permittivity is  $\mu_0$ . In the present example the conductors are thin wires along the  $z$  axis, and we suppose that the current density  $\mathbf{J}(\mathbf{r})$  is independent of azimuth in a cylindrical coordinate system  $(\rho, \phi, z)$  and is well approximated by a current  $I(z)$ . Then, the vector potential has only a  $z$  component,

$$A_z(\mathbf{r}) = \frac{\mu_0}{4\pi} \int I(z') \frac{e^{-ik\mathcal{R}}}{\mathcal{R}} dz'. \quad (3)$$

Since we work in the Lorenz gauge where

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} = 0, \quad (4)$$

the scalar potential  $V(\mathbf{r}, t) = V(\mathbf{r})e^{i\omega t}$  of the response fields is related to the vector potential according to

$$V(\mathbf{r}) = \frac{ic}{k} \frac{\partial A_z(\mathbf{r})}{\partial z} \equiv \frac{ic}{k} \partial_z A_z(\mathbf{r}). \quad (5)$$

The response fields  $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})e^{i\omega t}$  and  $\mathbf{B}(\mathbf{r}, t) = \mathbf{B}(\mathbf{r})e^{i\omega t}$  can then be calculated from the vector potential  $A_z(\mathbf{r})$  as

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}) - i\omega \mathbf{A}(\mathbf{r}) = -\frac{ic}{k} [\partial_{rz}^2 A_z(\mathbf{r}) \hat{\boldsymbol{\rho}} + (\partial_z^2 + k^2) A_z(\mathbf{r}) \hat{\mathbf{z}}], \quad (6)$$

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = -\partial_\rho A_z(\mathbf{r}) \hat{\boldsymbol{\phi}}. \quad (7)$$

The key relation between the incident electric field  $\mathbf{E}_{\text{in}}$  and the response field  $\mathbf{E}$  is that the tangential component of the total electric field  $\mathbf{E}_{\text{in}} + \mathbf{E}$  must vanish at the surface of the conductors. In the **thin-wire approximation** for wire radius  $a$  much less than the antenna half height  $h$ , the constraint is essentially on the  $z$ -component of the response electric field on the  $z$  axis,

$$E_z(0, 0, z) = -E_{\text{in}} = -E_0, \quad (8)$$

for the intervals  $[-h, -d/2]$  and  $[d/2, h]$  that contain the conductors of the antenna, where the gap between the terminals of the antenna has width  $d \ll h$ . From eq. (6), we obtain a differential equation for the vector potential on these intervals,

$$(\partial_z^2 + k^2) A_z(0, 0, z) = \frac{ik}{c} E_z(0, 0, z) = -\frac{ik}{c} E_0. \quad (9)$$

Two solutions to the homogeneous differential equation  $(\partial_z^2 + k^2) A_z(0, 0, z) = 0$  are, of course,  $\cos kz$  and  $\sin kz$ . A solution to the particular equation is simply the constant  $-iE_0/kc$ . Hence, a general solution to eq. (9) on the interval  $[d/2, h]$  can now be written as

$$A_z(0, 0, d/2 \leq z \leq h) = C_1 \cos kz + C_2 \sin kz - \frac{iE_0}{kc}. \quad (10)$$

since  $kd \ll 1$ . We expect that the vector potential will be symmetric about  $z = 0$ , so the solution on the interval  $[-h, -d/2]$  can be written as

$$A_z(0, 0, -h \leq z \leq -d/2) = C_1 \cos kz - C_2 \sin kz - \frac{iE_0}{kc}. \quad (11)$$

To evaluate the constants of integration  $C_1$  and  $C_2$  we need additional conditions on the system. In particular, we note that for a no-load (open circuit) receiving antenna, the current  $I(z)$  must vanish at the ends of the conductors, *i.e.*, at  $z = -h, -d/2, d/2$  and  $h$ . In the thin-wire approximation, the vector potential on the wire is proportional to the current in the wire at that point, because of the  $1/R$  dependence in eq. (3). In this approximation, the needed conditions on the vector potential are that it also vanishes at the ends of the conductors. From this we find,

$$C_1 = \frac{iE_0}{kc}, \quad C_2 = -\frac{iE_0}{kc} \frac{1 - \cos kh}{\sin kh}. \quad (12)$$

Finally, from eq. (5) we obtain the open-circuit voltage across the terminals,

$$\begin{aligned} V_{\text{oc}} &= V(0, 0, d/2) - V(0, 0, -d/2) = \frac{ic}{k}(A'_z(0, 0, d/2) - A'_z(0, 0, -d/2)) \\ &= 2icC_2 = -\frac{2E_0}{k} \frac{1 - \cos kh}{\sin kh}. \end{aligned} \quad (13)$$

For a short antenna with  $kh \ll 1$ , the open-circuit voltage is

$$V_{\text{oc}} = -E_0 h \quad (kh \ll 1), \quad (14)$$

in agreement with the estimate via dimensional analysis.

The effective height of the antenna is

$$H_{\text{eff}} = \left| \frac{V_{\text{oc}}}{E_0} \right| = \frac{\lambda}{\pi} \frac{1 - \cos(2\pi h/\lambda)}{\sin(2\pi h/\lambda)}. \quad (15)$$

### 3 Remarks

This example required a determination of the vector potential only at the antenna itself, and so is somewhat simpler than the task of determining the response fields in all space around the antenna. However the method used here is readily extended to a full solution of the antenna problem.

In particular, equations (3), (8) and (9) can be combined into an integral equation that relates the incident electric field at the conductors to the response currents in those conductors,

$$\int I(z')(\partial_z^2 + k^2) \frac{e^{-ik\mathcal{R}}}{\mathcal{R}} dz' = -\frac{4\pi ik}{Z_0} E_{\text{in}}(z), \quad (16)$$

where  $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377 \Omega$ . This is Pocklington's integral equation [4], whose solution is implemented numerically in codes such as NEC4 [13].

### 3.1 Short-Circuit Current in a Receiving Antenna

For the receiving antenna, it is of interest to calculate the current across its terminals when they are shorted. Then, using Thévenin's theorem [15], we could characterize the behavior of the antenna as part of the receiving circuit. An accurate calculation of the short-circuit current (or its equivalent, the antenna terminal impedance  $Z_A$ ) can/must be made by solving the integral equation (16).

Here, we illustrate the limitations of the thin-wire approximation in estimating the antenna impedance. When the antenna terminals are shorted, the constraint (8) on the response field that the total, tangential electric field vanish at the surface of the wire now applies over the entire interval  $[-h, h]$ . Then, a symmetric solution to the differential equation (9) for the vector potential on this interval is

$$A_z(0, 0, -h \leq z \leq h) = C \cos kz - \frac{iE_0}{kc}. \quad (17)$$

To determine the constant  $C$  we again require that the current, and hence the vector potential in the thin-wire approximation, vanish at the ends of the wire,  $z = \pm h$ , such that

$$A_z(0, 0, -h \leq z \leq h) = \frac{iE_0}{kc} \frac{\cos kz - \cos kh}{\cos kh}. \quad (18)$$

The thin-wire approximation to eq. (3) is that  $A_z(0, 0, z) \approx \mu_0 I(z)/4\pi$ , so we estimate that the short-circuit current at the terminals of the receiving antenna is

$$I_{sc} = \frac{4\pi}{\mu_0} \frac{iE_0}{kc} \frac{1 - \cos kh}{\cos kh} = \frac{4\pi iE_0}{kZ_0} \frac{1 - \cos kh}{\cos kh}, \quad (19)$$

where  $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377 \Omega$ .

The internal impedance  $Z_A$  of the antenna is then, according to Thévenin's analysis,

$$Z_A = \frac{V_{oc}}{I_{sc}} \approx -\frac{iZ_0}{2\pi} \cot kh. \quad (20)$$

This correctly indicates that the reactance of a short linear antenna is capacitive and that the reactance vanishes for  $h \approx \lambda/2$ , but the predicted divergence of the reactance for  $kh \ll 1$  is unphysical. Furthermore, the real part of the current, and also of the impedance, is neglected in the thin-wire approximation, so if the antenna were used as a transmitter, this analysis indicates that it would not consume any energy from the rf power source, *i.e.*, the antenna would not radiate.<sup>2</sup>

### 3.2 Transmitting Antenna

In the case of a transmitting antenna, the incident electric field is taken to be the internal field  $E_{in} = V_{in}/d$  of an rf generator that is located in the gap of width  $d$  between the terminals

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<sup>2</sup>The real part of the antenna impedance, the so-called radiation resistance  $R_{rad}$ , can be well calculated using the relation  $\langle P \rangle = I_0^2 R_{rad}/2$ , where  $\langle P \rangle$  is the time-average radiated power in the far zone as deduced from an appropriate thin-wire approximation to the current distribution caused by the rf power source. See, for example, [3].

of the antenna. This internal field is the negative of the (response) field  $E_z$  in the gap in the more realistic case that the rf generator is located some distance from the antenna and connected to it via a transmission line. The incident electric field is zero elsewhere on the conductors of the antenna. Then, the relation (8) can be extended for a center-fed linear dipole antenna to read

$$E_z(0, 0, z) = -E_{\text{in}}(z) = \begin{cases} 0 & (-h < z < -d/2), \\ -V_{\text{in}}/d & (-d/2 < z < d/2), \\ 0 & (d/2 < z < h). \end{cases} \quad (21)$$

However, use of the extreme form of the thin-wire approximation to solve eq. (16) for the transmitting antenna leads to response currents that imply response fields with nonzero  $E_z$  along the antenna conductors. While this approximation turns out to be good at predicting the response fields in the far zone, greater care is required for a good understanding of the response fields close to the antenna [16].

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