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The Poynting vector in Laguerre-Gaussian laser modes

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Abstract

It is found that the rotation of the Poynting vector of a Laguerre-Gaussian laser mode is proportional to the Gouy phase and for most cases of interest, rotates through less than one revolution in reaching the far field. For a $p=0$ mode, we show the expression for the rotation is particularly simple and independent of l. At the radius corresponding to maximum intensity, it yields a rotation of only $\pm\pi/2$ as $z\rightarrow\pm\infty$.

1. Introduction

Recently, there has been significant interest in the properties, generation and light matter interactions [1,2,3] of Laguerre-Gaussian laser modes. In the paraxial approximation the Laguerre-Gaussian modes, as with Hermite-Gaussian modes, satisfy Maxwell's equations. Both sets of modes are acceptable forms of the transverse distribution of the electric field within a laser cavity. In most lasers a slight departure from circular symmetry is sufficient to result in the eigenmodes of the oscillator being best described by the Hermite-Gaussian polynomials. Although lasers operating with transverse Laguerre-Gaussian modes have been reported [4], it is more usual to generate these Laguerre-Gaussian modes by the external conversion of a Hermite-Gaussian laser mode into the corresponding Laguerre-Gaussian mode. These so called mode converters can be based on cylindrical lenses [5,6] computer generated holograms [7] or spiral phaseplates [8].

It is usual to consider the flow of energy in an electromagnetic field in terms of the Poynting's theorem [9]. The theorem leads to a continuity equation or conservation law where the energy flow is represented by the Poynting vector given by $\boldsymbol{S}=\boldsymbol{E}\times\boldsymbol{H}$ which has the dimensions of energy per unit time per unit area. The electromagnetic linear momentum is given by $(\boldsymbol{E}\times\boldsymbol{B})/c^2$ and so in free space equals $\boldsymbol{S}/c^2$. In the original paper attributing orbital angular momentum to Laguerre-Gaussian laser modes [10], the linear momentum is calculated and the total angular momentum density $\epsilon_0\boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B})$, explicitly investigated.

It appears likely, therefore, that the detailed behaviour of the mode might be expressible in terms of the Poynting vector. Indeed, Fig. 2 of paper [10] is a "spiralling curve" which "represents the Poynting vector of a linearity polarised Laguerre-Gaussian mode". It has been shown that a torque may be applied on an atom by a near resonant Laguerre-Gaussian mode [1], leading to gross motion of the atom in rotation about the beam axis. The concept of a spiralling Poynting vector, $\boldsymbol{S}$, is thus immediately evocative. Unfortunately the multi-looped spiral displayed is, as we shall see, misleading; it only applies for a fixed radial position $\boldsymbol{r}$ in the beam as the mode propagates. When beam diver-

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gence is properly accounted for we demonstrate that the Poynting vector representing the path of the maximum intensity, which might be felt to be the best representation of the energy flow in the beam, does not spiral in this way.

2. Laguerre Gaussian modes

The field amplitude, E_{LG} Laguerre-Gaussian transverse laser modes are described by

$$E_{LG}(r, \phi, z) \propto \exp(-ikr^2/2R)\exp(-r^2/\omega^2) \times \exp[-i(2p+l+1)\psi]\exp(-il\phi) \times (-1)^p(r\sqrt{2}/\omega)^l \times L_p^l(2r^2/\omega^2) , \quad (1)$$

with

$$R(z) = (z_r^2+z^2)/z , \quad \omega^2(z) = 2(z_r^2+z^2)/kz_r , \quad \psi(z) = \arctan(z/z_r) , \quad (2)$$

where z is the distance from the beam waist, z_r is the Rayleigh range, k is the wave number, R is the radius of curvature of the wavefronts, ω is the radius at which the Gaussian term falls to $1/e$ of its on axis value, $(2p+l+1)\psi$ is the Gouy phase, r is the radius, ϕ is the azimuthal angle, p and l are the radial and azimuthal indexes of the mode respectively and $L_p^l(x)$ is the generalised Laguerre polynomial [11].

$$L_p^l(x) = (-1)^l \times \frac{d^l}{dx^l}[L_{l+p}(x)] . \quad (3)$$

The azimuthal phase term, $\exp(-il\phi)$, distinguishes the Laguerre-Gaussian from the Hermite-Gaussian modes. It has been shown theoretically that this term gives rise to a well defined angular momentum of $l\hbar$ per photon independent of the $\pm\hbar$ associated with the polarisation state [10]. This is termed the orbital angular momentum. Although orbital angular momentum is quantised it is, of course, not exclusively a photon property but is describable in terms of the amplitude and phase of the electric field distribution within the beam, in the same way as polarisation. The azimuthal phase dependence of these modes can best be visualised by examining the interference pattern between a Laguerre-Gaussian mode and a plane wave of the same frequency [12] (see Fig. 1).

For a Hermite-Gaussian mode, the surfaces of constant phase are a series of discs separated by the wavelength of the light. By contrast, for a Laguerre-Gaussian mode with $l \neq 0$, the surfaces of constant phase have

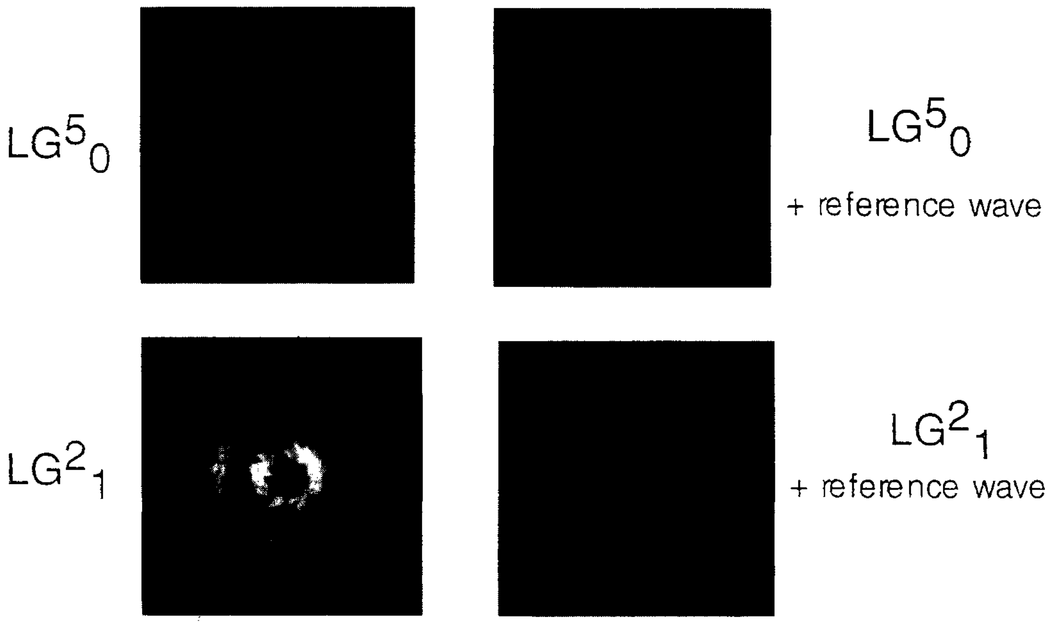

Fig. 1. The intensity distribution of two different Laguerre-Gaussian laser modes and the interference pattern between the Laguerre-Gaussian modes and a near plane-wave reference wave of the same frequency.

helical form [13] and the resulting on-axis phase discontinuity is consistent with the zero on-axis intensity of these modes. The helical nature of the phase surface implies that, even in free space, the wavefronts are not perpendicular to the direction of propagation of the beam and so the Poynting vector is not parallel to the direction of propagation. The Poynting vector has a magnitude of energy per second per unit area and a direction which represents the energy flow at any point in the field. In this discussion we are concerned only of the direction of the Poynting vector, **S**, which for a linearly polarised Laguerre-Gaussian is

$$\bar{S}=C\frac{z_r}{z_r^2+z^2}\left[\frac{zr}{z_r^2+z^2}\hat{r}+\frac{l}{kr}\hat{\boldsymbol{\phi}}+\left(1+\frac{r(z_r^2-z^2)}{2(z_r^2+z^2)^2}+\frac{(2p+l+1)z_r}{(z_r^2+z^2)k}\right)\hat{z}\right], \quad (4)$$

where C is a constant which depends on the radial position within the intensity distribution, the wavelength of the light and is proportional to the total power in the beam. If small terms in the $\hat{z}$ coefficient are ignored, this simplifies to [10]

$$\bar{S}=C\frac{z_r}{z_r^2+z^2}\left(\frac{zr}{z_r^2+z^2}\hat{r}+\frac{l}{kr}\hat{\boldsymbol{\phi}}+\hat{z}\right). \quad (5)$$

The $\hat{\boldsymbol{\phi}}$ component in Eq. (5) implies that Poynting vector has an azimuthal component as it propagates away from the beam waist. From Eq. (5), the rate of the azimuthal rotation, ϑ, is related to the distance, z, away from the beam waist by

$$\partial\vartheta/\partial z=l/kr(z)^2. \quad (6)$$

From Eq. (6) we see that, for a constant radius, $\boldsymbol{E}\times\boldsymbol{H}|_r$ follows a spiral path with a well defined pitch, z_p, given by

$$z_p=2\pi kr^2/l. \quad (7)$$

However, the presence of a radial term in Eq. (5), which arises from the divergence of the beam, means that a constant radius does not correspond to a particular point in the relative intensity distribution. Consequently Eq. (7) does not represent the path of the energy flow of the beam. We must chose a radius which corresponds to a fixed point within the relative intensity distribution, such that $r(z)/\omega(z)$ is a constant, using Eq. (2), we can then write Eq. (6) as

$$\frac{\partial\vartheta}{\partial z}=\frac{l}{2}\left(\frac{\omega(z)}{r(z)}\right)^2\frac{z_r}{z_r^2+z^2}. \quad (8)$$

The total rotation, ϑ, of the Poynting vector from the beam waist to a position z, can be simply obtained by integrating Eq. (8) with respect to z as

$$\vartheta=\frac{l}{2}\left(\frac{\omega(z)}{r(z)}\right)^2\arctan\left(\frac{z}{z_r}\right). \quad (9)$$

We note that the rotation of the Poynting vector is proportional to the change in the Gouy phase from that at the beam waist $z=0$. Clearly, the extent of the rotation of the Poynting vector depends on the value of $r(z)/\omega(z)$. A representative radius for the main flow of energy is that corresponding to the maximum field of the Laguerre-Gaussian laser mode.

The field distribution of a Laguerre-Gaussian mode with $p=0$ is a single ring with a $2\pi l$ azimuthal phase variation. The radius of maximum field amplitude, $r_{\mathrm{E-Max}}(z)$, is trivially,

$$r_{\mathrm{E-Max}}(z)=\frac{\sqrt{2}\,\omega(z)}{2}\sqrt{l}. \quad (10)$$

Substitution of this radius into Eq. (8), gives the rotation of the Poynting vector associated with the radius corresponding to the maximum field of a $p=0$ mode $(l\neq 0)$ as

$$\vartheta=\arctan(z/z_r), \quad (11)$$

which surprisingly is independent of l.

For a $p=0$ mode, Eq. (11) shows that far from describing a multi-turned spiral path [10], the Poynting vector corresponding to the maximum field rotates by exactly $\pi/2$ either side of the beam waist as it propagates to the far field and that at the Rayleigh range the rotation is $\pm\pi/4$. At all points, the rotation is proportional to the Gouy phase with a constant of proportionality of $(1+l)^{-1}$.

From the substitution of Eq. (10) into Eq. (5) the Poynting vector for the radius of maximum field is,

$$\bar{S}=C\frac{z_r}{z_r^2+z^2}\left(\sqrt{\frac{lz^2}{kz_r(z_r^2+z^2)}}\hat{r}+\sqrt{\frac{lz_r}{k(z_r^2+z^2)}}\hat{\boldsymbol{\phi}}+\hat{z}\right). \quad (12)$$

We see that in the far field, $\hat{\boldsymbol{\phi}}$ tends to zero and hence the Poynting vector is no longer rotating while at the

Raleigh range the $\hat{\phi}$ and $\hat{r}$ coefficients are of equal magnitude.

For the Laguerre-Gaussian modes with $p \neq 0$, the situation is more complex as these modes have a field distribution of $(p+1)$ concentric rings and consequently have a number of local maxima in the electric field. The $p=1$ modes, for example, have a field amplitude distribution is given by

$$E_{LG,p=1}(r) \propto \exp(-r^2/\omega^2) \times (r\sqrt{2}/\omega)^l \times [l+1-(2r^2/\omega^2)] \,. \tag{13}$$

Following the same procedure as before, the total rotation of the Poynting vector, from the beam waist to a position z, for the inner and outer rings of a $p=1$ mode are given by

$$\vartheta_{\rm inner} = \frac{2l}{2l+3-\sqrt{8l+9}} \arctan(z/z_r) \,,$$

$$\vartheta_{\rm outer} = \frac{2l}{2l+3+\sqrt{8l+9}} \arctan(z/z_r) \,. \tag{14}$$

For $p>1$ the analytical solutions for the radii of maximum field become unwieldy, but the numerical values of the $(l/2)[\omega(z)/r(z)]^2$ term in Eq. (9) can readily be calculated for a variety of modes and are shown in Table 1. Note that in all cases it is the Poynting vector associated with the local field maximum of the inner ring which undergoes the largest rotation. The combination of high p with low l results in the largest rotation of the Poynting vector, but it is only for $p>2$ that the rotation can exceed a complete revolution. For example, Fig. 2 shows the propagation of the Poynting vector, which we have calculated numerically, associated with the various rings of the Laguerre-Gaussian mode of index $p=3$ and $l=1$. For a particular mode, the rotation is once again proportional to the Gouy phase but the constant of proportionality is now more complicated.

3. Conclusions

Although our calculation is essentially trivial, it gives a surprising and potentially useful result. We have shown that the total rotation of the Poynting vector associated with the radius corresponding to peak intensity as it propagates from the beam waist to the far field, is proportional to the Gouy phase and of the order of one complete revolution or less for experimentally accessible values of p with low l. It would therefore appear that there is no link, for example, between the predicted rotation imposed on a atom by a near resonant Laguerre-Gaussian mode with the rotation of the Poynting vector. The orbital angular momentum responsible for the rotating atom arises from the azimuthal phase, while the rotation of the Poynting vector depends upon the Gouy phase. These modes are not only of interest in atom-field interactions. Interactions with non-resonant dielectric media are also of great interest [14] and the fullest understanding of their

Table 1
The numerical values of $(l/2)(\omega(z)/r(z))^2$ for the radius corresponding to the local maximum amplitudes of various Hermite-Gaussian modes

	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$
$p=0$	1	1	1	1	1	1
$p=1$						
inner ring	2.281	2.000	1.843	1.740	1.667	1.611
outer ring	0.219	0.333	0.407	0.460	0.500	0.532
$p=2$						
inner ring	3.480	2.902	2.576	2.368	2.221	2.111
2nd ring	0.388	0.553	0.654	0.716	0.759	0.791
outer ring	0.123	0.205	0.266	0.314	0.353	0.385
$p=3$						
inner ring	4.676	3.785	3.292	2.974	2.746	2.576
2nd ring	0.539	0.754	0.865	0.929	0.972	1.000
3rd ring	0.194	0.315	0.398	0.459	0.506	0.544
outer ring	0.085	0.148	0.199	0.220	0.275	0.305

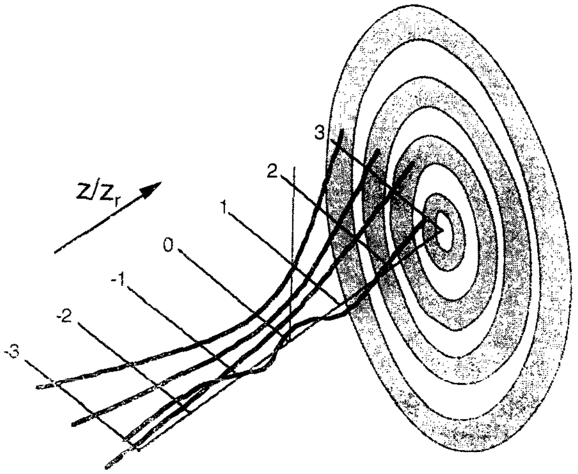

Fig. 2. The propagation of the Poynting vector associated with the various rings of the Laguerre-Gaussian mode $p=3$, $l=1$.

properties is required. Whether interactions directly reflecting the dependence on the Gouy phase and the associated rotation of the Poynting vector will be found remains to be seen. We have found that the Poynting vector of a Laguerre-Gaussian mode behaves in an unusual and non-intuitively obvious manner.

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