version of amplitudes between first and second extremes. The measured value of the rise time Δt_r is (16 ± 4) ns and through Eq. (8) one gets a fall time for the corresponding $V_s(t)$ of (20 ± 6) ns, which is in good agreement with the value measured from Fig. 2.

FINAL REMARKS

The effect of the switching action of a real electrical switch on simple circuits can be predicted, with moderate computational effort, yielding a good description of the measured behavior.

The subject can be presented to the students not only theoretically but also experimentally, using low voltages and semiconductor devices as switches. However, care must be taken in the design of these experiments because of the large percentage of energy that may be dissipated in the switch.

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Derivation of reciprocity relations for a beam splitter from energy balance

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It is shown that the usual amplitude and phase relations connecting the reflectance and transmittance of a stratified or continuous, nonabsorbing beam splitter can be derived by a simple energy balance argument relating to a Michelson interferometer.

I. INTRODUCTION

Reciprocity relations connecting the complex reflectance r and transmittance t of a nonabsorbing beam splitter with the corresponding quantities r',t' for light incident from the opposite direction have been well known in optics for a long time.[1,2] The subject has nevertheless received renewed attention in recent years, as the original reciprocity relations of Stokes were generalized.[3–5] In particular, for a beam splitter in the form of a stratified multilayer, where the information about r,t,r',t' is contained in the so-called characteristic matrix,[1,2] the expressions for r,t,r',t' become quite complicated. However, the following relations have been derived for an arbitrary angle of incidence,

$$|r| = |r'|, \quad |t| = |t'|, \tag{1}$$

$$|r|^2 + |t|^2 = 1 = |r'|^2 + |t'|^2, \tag{2}$$

$$r^*t' + t^*r' = 0, \tag{3}$$

in some cases after "lengthy...calculation."[3] Equation (2) obviously expresses the conservation of energy at the beam splitter, but Eq. (3) has not usually been interpreted in the same way. The relations have been derived by use of the principle of time reversal invariance.[4]

We would like to draw attention to the fact that the three relations (1)–(3) can all be derived by a simple energy balance argument relating to a Michelson interferometer. It applies to any beam splitter in which a plane wave incident from one side emerges from the other side as a plane wave traveling in the same direction. This article contains no new results, but only a simplified derivation of some previously derived results that holds under rather general conditions and provides some additional insight.

II. DERIVATION

We consider the Michelson type of interferometer system illustrated in Fig. 1, containing a parallel-sided beam splitter whose refractive index is a function only of the coordinate perpendicular to the face, either discrete or continuous. The beam splitter may therefore be a stratified or a continuous medium. To ensure that a plane wave incident from one side emerges as a plane wave traveling in the same direction, we shall suppose that the same medium (e.g., air) is in contact with both sides. Let a linearly polarized (TE or TM) monochromatic plane wave, of unit complex amplitude referred to point A, be incident from the left at some arbitrary angle, as shown, and let ϕ_0 be the phase shft suffered by this wave in propagating from A to P. The beam splitter then gives rise to reflected and transmitted waves of complex amplitudes r and t referred to points P and Q, respectively, which travel to two perfect mirrors $M1$ and $M2$, where they are reflected straight back. Let ϕ_1 and ϕ_2 be the (arbitrary) phase shifts experienced by these waves along the paths PC + CP and QB + BQ, respectively. The two returning waves both give rise to further reflected and transmitted components, and we shall use r',t' to denote the complex reflectance and transmittance of the beam splitter for waves incident from the right. As a result, outgoing waves with complex amplitudes E_A, E_D emerge at A and D. Let ϕ_3 be the phase shift corresponding to the path from Q to D.

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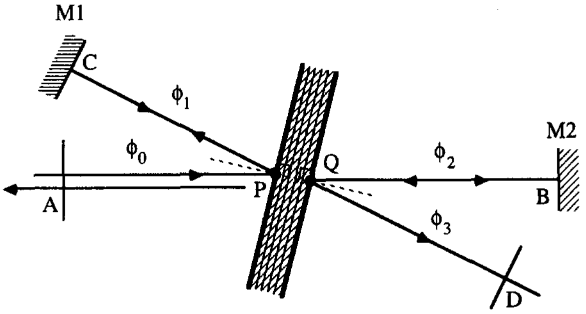

Fig. 1. Outline of the geometry for the Michelson interferometer.

Reference to Fig. 1 shows that we may represent E_A and E_D by the combinations

$$E_A = e^{i\phi_0} r e^{i\phi_1} r e^{i\phi_0} + e^{i\phi_0} t e^{i\phi_2} t' e^{i\phi_0}$$
$$= e^{2i\phi_0}(r^2 e^{i\phi_1} + tt' e^{i\phi_2}), \tag{4}$$
$$E_D = e^{i\phi_0} r e^{i\phi_1} t e^{i\phi_3} + e^{i\phi_0} t e^{i\phi_2} r' e^{i\phi_3}$$
$$= e^{i(\phi_0 + \phi_3)}(rte^{i\phi_1} + tr' e^{i\phi_2}). \tag{5}$$

From Eqs. (4) and (5) we immediately find for the corresponding light intensities,

$$I_A = |E_A|^2 = |r|^4 + |t|^2|t'|^2 + 2|r^2 tt'|\cos(\theta_t + \theta_t' - 2\theta_r + \phi_2 - \phi_1), \tag{6}$$

$$I_D = |E_D|^2 = |rt|^2 + |r't|^2 + 2|rr't^2| \times\cos(\theta_r' - \theta_r + \phi_2 - \phi_1), \tag{7}$$

where we have written

$$r = |r|e^{i\theta_r}, \quad r' = |r'|e^{i\theta_r'},$$
$$t = |t|e^{i\theta_t}, \quad t' = |t'|e^{i\theta_t'}. \tag{8}$$

Now, if energy is to be conserved, the sum of the outgoing light intensities I_A, I_D must equal unity, which is the incoming light intensity. Hence,

$$1 = |r|^2(|r|^2 + |t|^2) + |t|^2(|t'|^2 + |r'|^2) + 2|r^2 tt'|\cos(\theta_t + \theta_t' - 2\theta_r + \phi_2 - \phi_1) + 2|rr't^2|\cos(\theta_r' - \theta_r + \phi_2 - \phi_1). \tag{9}$$

Now the third and fourth terms on the right contain the arbitrary phase shift $\phi_2 - \phi_1$. As the right-hand side must equal unity and these two terms cannot be made separately equal to zero, they must sum to zero, while the first two terms sum to unity. This requires the amplitudes of the two cosine terms to be made equal, so that

$$|r||t'| = |r'||t|, \tag{10}$$

while

$$|r|^2 + |t|^2 = 1 = |r'|^2 + |t'|^2. \tag{11}$$

From Eqs. (10) and (11),

$$|r|/|t| = |r'|/|t'|$$

or

$$|r|/\sqrt{1 - |r|^2} = |r'|/\sqrt{1 - |r'|^2}$$

or

$$|r| = |r'|, \quad |t| = |t'|. \tag{12}$$

Hence, on combining the two cosine terms in Eq. (9), we obtain

$$1 = 1 + 4|r|^2|t|^2 \cos\left(-\frac{3\theta_r}{2} + \frac{\theta_r'}{2} + \frac{\theta_t}{2} + \frac{\theta_t'}{2} + \phi_2 - \phi_1 \right)\cos\left(\frac{\theta_t - \theta_r + \theta_t' - \theta_r'}{2}\right). \tag{13}$$

Now the second term has to vanish. But, because the first cosine factor in this term contains the arbitrary phase difference $\phi_2 - \phi_1$, it is the second cosine factor that must vanish in general. Hence, we conclude that

$$\theta_t - \theta_r + \theta_t' - \theta_r' = \pm\pi \tag{14}$$

in all cases. For a symmetric beam splitter, this would, of course, imply that $\theta_t - \theta_r = \pm\pi/2$. From Eq. (14),

$$e^{i(\theta_t' - \theta_r)} + e^{i(\theta_r' - \theta_t)} = 0, \tag{15}$$

and on multiplying both sides of this equation by $|r||t|$ and using Eq. (12) we obtain

$$r^* t' + t^* r' = 0. \tag{16}$$

III. CONCLUSION

We have derived the three reciprocity relations (1)–(3), and we see that all three of them follow from considerations of energy balance alone. Moreover, it makes no difference if the medium is stratified in several layers or continuous, so long as the index of refraction depends on only one coordinate. That the phase relationship (14) is required by energy balance just as the more obvious energy relations (11) appears not to have been generally recognized.

ACKNOWLEDGMENTS

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