# TM and TE electromagnetic beams in free space

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Received May 12, 1980

The existence of unguided TM and TE Gaussian beams is derived in a simple way from the wave equation for the vector potential, and the field configurations are discussed. We point out the possibility of a new type of free-electron laser.

In a previous work,[1] a simple method was described for obtaining the electromagnetic fields of the so-called $TEM_{00}$ mode for unguided beams. It was assumed that the magnetic-vector potential is plane polarized in the transverse direction, and in Cartesian coordinates the wave equation for the potential was solved, yielding expressions for the **E** and **H** fields. Some field lines of this mode are depicted in Fig. 1(a). This Letter is a continuation of that work; we now assume that the vector potential is polarized in the *axial* direction. Using cylindrical coordinates, our approach immediately gives the electromagnetic fields of TM Gaussian–Laguerre modes, and, by using duality, the corresponding TE modes are found as well. The existence of Gaussian–Laguerre modes of radiation in open structures was first discussed by Goubau and Schwering.[2] They describe an electromagnetic beam by a continuous spectrum of cylindrical-wave solutions of Maxwell's equations. The motivation for their selection of the cylindrical-wave spectral amplitudes for **E** and **H** is rather obscure, and they do not investigate the field configurations in detail because they were not interested in that aspect of the problem. In the present work, the TM and TE unguided modes appear naturally as solutions of the wave equation. Our main purpose is to present a simple and concise method for calculating the field properties of pure TM and TE unguided electromagnetic-beam modes and to work out the field lines for the lowest-order case. A second objective is to point out the possibility for coherent amplification of an unguided TM wave by interaction of its longitudinal electric-field component with an axial beam of free electrons.

In the approach of Ref. 1, one first solves the wave equation for the magnetic-vector potential **A**, which, for a monochromatic wave in free space, and using the Lorentz gauge, is (time dependence $e^{i\omega t}$ understood)

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = 0. \tag{1}$$

In this work, instead of assuming that **A** is purely transverse, we consider solutions in which **A** is purely longitudinal and of the form

$$A_z = \psi e^{-ikz}, \tag{2}$$

where $\psi$ is a slowly varying function of $z$. The approximate lowest-order solution is well known[3] to be

$$\psi = iQe^{-iQ\rho^2}, \quad \rho^2 = \frac{x^2+y^2}{w_0{}^2}, \tag{3a}$$

$$iQ = \frac{1}{1-iz/l}, \tag{3b}$$

in which $w_0$ denotes the spot size at the beam waist and one defines the diffraction length parameter $l = \pi w_0{}^2/\lambda$. The paraxial approximation used here is valid so long as $w_0 \gg \lambda$. If needed, corrections to the paraxial solution may be obtained as in Ref. 1.

The magnetic field $\overline{H} = \nabla \times \mathbf{A}$ is seen to be purely transverse when **A** is axial. In cylindrical coordinates $(r,\phi,z)$, for the fundamental mode one finds that

$$H_\phi = \frac{2iQ\rho}{w_0} A_z \tag{4}$$

and $H_r = H_z = 0$. It is useful to write

$$iQ = \frac{w_0}{w} e^{i\Phi}, \tag{5}$$

where $\Phi = \tan^{-1}(z/l)$ represents a beam phase shift and $w(z) = w_0\,[1+(z/l)^2]^{1/2}$ is the beam contour. The electric field of this TM mode may found from

$$\mathbf{E} = -\frac{i}{k} \nabla \times \mathbf{H}. \tag{6}$$

One finds that

$$E_r = H_\phi, \tag{7a}$$

$$E_\phi = 0, \tag{7b}$$

$$E_z = \frac{2Q}{l}(1 - iQ\rho^2) A_z, \tag{7c}$$

where in obtaining Eq. (7a) the approximation is made:

$$\frac{\partial A_z}{\partial z} = -ik\,A_z.$$

It is evident that the magnetic-field lines are circles concentric with the beam ($z$) axis. Some field lines, in the region of the beam waist, are suggested in Fig. 1(b). The equations for the electric-field lines may be calculated using the method described by Lorrain and Corson[4] for a radiating electric dipole. One obtains, for the equation of an electric-field line of a traveling wave,

$$\frac{r^2}{w^2} e^{-r^2/w^2} \cos\left(\omega t - kz - \frac{kr^2}{2R} + 2\Phi\right) = K,$$

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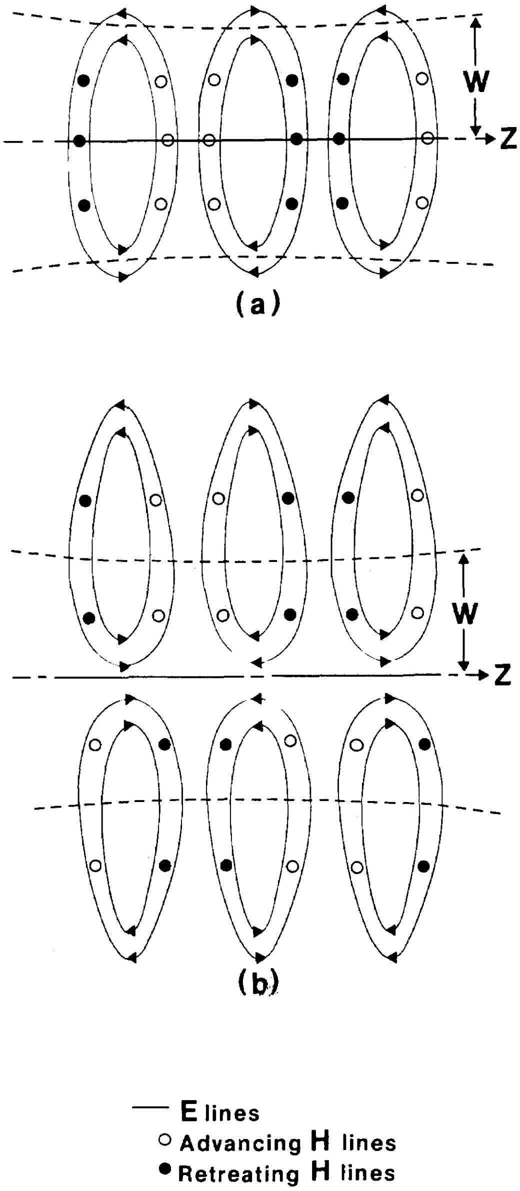


Fig. 1. Field distributions of unguided electromagnetic beams in a longitudinal section passing through $z$ axis. (a) The $TEM_{00}$ mode and (b) the fundamental TM mode. The ratio $\lambda/w_0$ normally would be much smaller than depicted here. Note that along the axis ($r = 0$), the longitudinal electric field of the $TEM_{00}$ mode vanishes, whereas that of the lowest-order TM mode has maximum amplitude.

where $R = z[1 + (l/z)^2]$ denotes the radius of the $A_z$ phase fronts[5] and $K$ is constant for a given field line. For a standing wave, an electric-field line obeys

$$\frac{r^2}{w^2} e^{-r^2/w^2} \cos \omega t \cos\left(kz + \frac{kr^2}{zR} - 2\Phi\right) = K.$$

Near the beam axis, the field lines of the unguided fundamental TM mode strongly resemble those of a $TM_{01}$ mode propagating inside an ideal circular cylindrical conductor. In particular, the lowest-order free TM mode has a $z$ component of electric field along the beam axis.[6] The existence of a longitudinal electric-field component presents a mechanism whereby charges can interact with the electromagnetic field of the mode. Applications of this situation are well known in the microwave range, in which the guided waves of linear accelerators deliver energy to electron beams. The idea of using an optical laser cavity as an electron accelerator was apparently first proposed by Shimoda.[7] It occurs to us that it might be possible to reverse the process and use an electron beam to amplify or generate TM beams of radiation in open structures at millimeter and possibly at shorter wavelengths. This possibility, which, if implemented, would lead to lasers quite different from the present free-electron lasers, is now under study.

Higher-order solutions of the wave equation yield for the TM modes

$$A_z = g\left(\frac{r}{w}\right) \cos n\theta \frac{w_0}{w} \exp[i(\Phi - Kz - Q\rho^2)], \quad (8a)$$

where

$$g\left(\frac{r}{w}\right) = \left(\sqrt{2}\frac{r}{w}\right)^n L_p{}^n \left(2\frac{r^2}{w^2}\right), \quad (8b)$$

in which $L_p{}^n$ denotes a generalized Laguerre polynomial and $n$ and $p$ are angular and radial mode numbers. The phase shift is now given by

$$\Phi\,(p, n; z) = (2p + n + 1)\arctan(z/l). \quad (8c)$$

Finally, we note that, in free space, the principle of duality applies. If one obtains any solution of Maxwell's equations, then another may be obtained by replacing **E** with **H** and **H** with −**E**. Thus for every unguided TM mode there is a corresponding TE mode.

## References

1. L. W. Davis, "Theory of electromagnetic beams," Phys. Rev. A **19**, 1177 (1979).
2. G. Goubau and F. Schwering, "On the guided propagation of electromagnetic wave beams," IRE Trans. Antennas Propag. **AP-9**, 248 (1961).
3. H. Kogelnik and T. Li, "Laser beams and resonators," Proc. IEEE **54**, 1312 (1966).
4. P. Lorrain and D. R. Corson, *Electromagnetic Fields and Waves*, 2nd ed. (Freeman, San Francisco, 1970).
5. In the paraxial approximation, the equation for the phase fronts of $A_z$ is $z = \text{const.} - r^2/2R$.
6. Simple considerations about the nature of field lines make it obvious that any wave of finite lateral extent has longitudinal field components. In the case of the so-called $TEM_{00}$ mode, it is seen that the longitudinal electric field vanishes along the $z$ axis.
7. K. Shimoda, "Proposal for an electron accelerator using an optical maser," Appl. Opt. 1, 33 (1962).