Confocal Multimode Resonator for Millimeter Through Optical Wavelength Masers

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Multimode resonators of high quality factor will very likely play a significant role in the development of devices, such as the maser, which operate in the millimeter through optical wavelength range. It has been suggested that a plane-parallel Fabry-Perot interferometer could act as a suitable resonator. In this paper a resonator consisting of two identical concave spherical reflectors, separated by any distance up to twice their common radius of curvature, is considered.

Mode patterns and diffraction losses for the low-loss modes of such a resonator are obtained analytically, using an approximate method which was suggested by W. D. Lewis. The results show that the diffraction losses are generally considerably lower for the curved surfaces than for the plane surfaces. Diffraction losses and mode volume are a minimum when the reflector spacing equals the common radius of curvature of the reflectors. For this case the resonator may be termed confocal. A further property of the concave spherical resonator is that the optical alignment is not extremely critical.

I. INTRODUCTION

Schawlow and Townes[1] proposed that coherent amplification could be achieved in the infrared through optical regions of the frequency spectrum by maser techniques. At such frequencies multimode resonators are necessary to achieve reasonable dimensions and high Q. They and Prokhorov[2] and Dicke[3] have suggested as a resonator two plane-parallel reflecting planes, known as a Fabry-Perot interferometer, or etalon.[4]

In Fabry-Perot resonators the major factors contributing to the Q (i.e., resolving power) are reflection losses and diffraction losses. Reflection losses result from absorption in the reflectors, and from transmission through them. At optical frequencies a very good layered dielectric reflector[5] can have a $99\frac{1}{2}$ per cent reflection coefficient. Diffraction losses result from the finite aperture of the reflectors and from imperfections in their "flatness."

Fox and Li have shown in the accompanying paper[6] that modes, in the sense of a self-reproducing field pattern, exist for an open structure such as a Fabry-Perot interferometer. They also have recognized that the diffraction losses of a plane-parallel Fabry-Perot are very much less than those obtained by assuming a uniform intensity distribution over the reflector and the Fraunhofer far field diffraction angle. They have made numerical self-consistent field calculations based on Huygens' principle to determine the actual diffraction losses and mode patterns.

In interferometry using a Fabry-Perot resonator, one normally excites a system of plane waves traveling at certain discrete angles to the axis. Constructive interference at each of these discrete angles, as is appropriate to ring order, wavelength and spacing, results in a pattern of concentric bright rings. Schawlow and Townes indicated that each ring of the interference pattern is *not* a pure mode of the resonator but an infinite sum of such modes, each representing a different field pattern over the reflector. This idea has been given much substance by the work of Fox and Li.

The plane-parallel Fabry-Perot is not necessarily ideal, however, as a high-frequency multimode resonator. A resonator formed by two spherical reflectors of equal curvature separated by their common radius of curvature is considered in detail in this paper. The focal length of a spherical mirror is one-half of its radius of curvature. Therefore the focal points of the reflectors are coincident and the resonator is termed confocal. G. W. Series, Fox and Li[6] and Lewis[7] have also suggested the confocal resonator. Lewis has recognized that it would have lower diffraction losses than the plane-parallel Fabry-Perot and has described the analytic solution presented here.

The use of confocal reflectors as an interferometer has been described by Connes.[8] The adjustment of the spherical Connes interferometer is trivial compared to the Fabry-Perot. Parallelism between the reflectors is not a strict requirement, the only fine adjustment therefore being the spacing between the surfaces. Parabolic surfaces may also be used, but they have an axis and thus lose the advantage of ease of adjustment.

II. RESONATOR QUALITY FACTOR

Resonator quality factor, or Q, is defined as

$$Q = \omega \frac{\text{energy stored}}{\text{energy lost per second}}. \tag{1}$$

Consider an interferometer consisting of two reflecting surfaces separated by a distance d which is large compared to the wavelength in the medium λ. By considering waves bouncing back and forth between the surfaces, one may derive an approximate Q as

$$Q = \frac{2\pi d}{\alpha\lambda}, \tag{2}$$

where α is the fractional power loss per bounce from a reflector and is the sum of diffraction and reflection losses. This is to be compared to the resolving power derived in optics[9] as

$$R = \frac{2\pi d\sqrt{r}}{\lambda(1 - r)}, \tag{3}$$

where the power reflection coefficient per bounce is $r \equiv 1 - \alpha$. Resolving power is thus synonymous with Q within the small loss approximation of (2).

If diffraction losses are small compared with reflection losses, then resonator Q is proportional to the spacing between the reflecting surfaces. For a given reflector aperture size, the resonator Q will continue to increase with the spacing d between the reflectors until the diffraction losses become roughly comparable with the reflection losses. Further increase in spacing then decreases the Q because of increasing diffraction losses.

III. MODES AND DIFFRACTION LOSSES OF A CONFOCAL RESONATOR

All resonator dimensions are assumed large compared to a wavelength; the modes and diffraction losses of the confocal resonator are therefore obtainable from a self-consistent field analysis using Huygens' principle.[10] A confocal resonator is considered, with identical spherical reflectors of radius b, as shown in Fig. 1. Assume the field to be linearly polarized over the P' surface in the y direction and given by $E_0 f_m(x')g_n(y')$, where E_0 is a constant amplitude factor and $f_m(x')$ and $g_n(y')$ are the field variations over the aperture. At point $P(x,y)$ on the other surface, one computes the electric field by summing over contributions from the differential Huygens sources at all points $P'(x',y')$. The result is

$$E_y = \int_{S'} \frac{ik(1 + \cos\theta)}{4\pi\rho} e^{-ik\rho} E_0 f_m(x')g_n(y')\, dS'. \tag{4}$$

Here ρ is the distance between P and P', θ is the angle between the line $P'P$ and the normal to the reflector surface at P', and k is the propagation constant of the medium between the reflectors. Note that $k =$

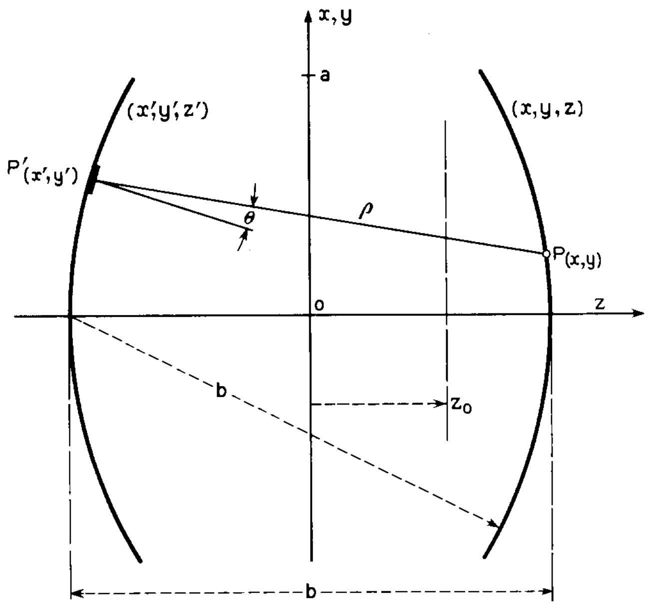

Fig. 1 — Confocal resonator with spherical reflectors.

$2\pi/\lambda$, where λ is the wavelength in the medium. The electric field in the xz plane is approximately zero. The reflector is assumed *square* and of dimension $2a$, which is small compared to the spacing b (since the confocal spacing is under consideration $d = b$), and thus θ is very nearly zero. The medium is assumed to fill all space.

The normal modes or eigenfunctions of the confocal resonator are obtained by requiring that the field distribution over $x'y'$ reproduce itself within a constant over the xy aperture, and thus $E_y = E_1 f_m(x) g_n(y)$, where $E_1 = \sigma_m \sigma_n E_0$. The proportionality factor $\sigma_m \sigma_n$ is generally complex, giving both amplitude and phase changes. The resulting integral equation is

$$\sigma_m \sigma_n f_m(x) g_n(y) = \iint_{-a}^{+a} \frac{ik}{2\pi\rho} e^{-ik\rho} f_m(x') g_n(y') \, dx' dy' . \tag{5}$$

The distance ρ varies only a small amount for small apertures and thus may be replaced by the separation b except in the exponential phase term. For x and y small compared to b one can show that

$$\frac{\rho}{b} = 1 - \frac{xx' + yy'}{b^2} + \frac{w^2 w'^2}{4b^4} + \cdots , \tag{6}$$

where $w^2 = x^2 + y^2$. The third term makes a negligible contribution to the phase when $a^2/b\lambda \ll b^2/a^2$. Note that in this approximation one cannot distinguish between spherical and parabolic surfaces. In terms of some dimensionless variables

$$c \equiv \frac{a^2 k}{b} = 2\pi\left(\frac{a^2}{b\lambda}\right) \qquad X \equiv \frac{x\sqrt{c}}{a}, \qquad Y \equiv \frac{y\sqrt{c}}{a}, \tag{7}$$

and with $F_m(X) \equiv f_m(x)$ etc., (5) becomes

$$\sigma_m \sigma_n F_m(X) G_n(Y) = \frac{ie^{-ikb}}{2\pi} \int_{-\sqrt{c}}^{+\sqrt{c}} F_m(X') e^{+iXX'}\, dX' \cdot \int_{-\sqrt{c}}^{+\sqrt{c}} G_n(Y') e^{+iYY'}\, dY'. \tag{8}$$

Slepian and Pollak[11] have considered the following integral equation:

$$F_m(X) = \frac{1}{\sqrt{2\pi}\chi_m} \int_{-\sqrt{c}}^{+\sqrt{c}} F_m(X') e^{+iXX'}\, dX'. \tag{9}$$

This is a homogeneous Fredholm equation of the second kind with $e^{iXX'}$ as the kernel. It is often referred to as a finite Fourier transform. They have shown solutions to be

$$F_m(c,\eta) \propto S_{0m}(c,\eta), \tag{10}$$

$$\chi_m = \sqrt{\frac{2c}{\pi}}\, i^m R_{0m}^{(1)}(c,1), \qquad m = 0, 1, 2, \cdots, \tag{11}$$

where $S_{0m}(c,\eta)$ and $R_{0m}^{(1)}(c,1)$ are respectively the angular and radial wave functions in prolate spheroidal coordinates as defined by Flammer,[12] and where $\eta = X/\sqrt{c} = x/a$ and $\eta = Y/\sqrt{c} = y/a$ respectively for $F_m(X)$ and $G_n(Y)$. There is an infinite number of eigenfunctions and corresponding eigenvalue solutions to (9) for any value of c. Flammer[12] gives values of these functions for $c \leqq 5$ and Slepian and Pollak[11,13] have computed the eigenvalues χ_m for the important region of $c > 5$.

The eigenfunction solutions of (8) are thus the spheroidal wave functions $S_{0m}(c,x/a)S_{0n}(c,y/a)$. The eigenfunctions are real; therefore, the reflecting surfaces are of constant phase. The eigenvalues are

$$\sigma_m \sigma_n = \chi_m \chi_n i\, e^{-ikb}. \tag{12}$$

The phase shift between the two reflecting confocal surfaces equals the phase angle of $\sigma_m \sigma_n$. For resonance the round-trip phase shift must equal an integer q times 2π. From (11) and (12), one finds therefore

$$2\pi q = 2 \left| \frac{\pi}{2} - kb + (m + n) \frac{\pi}{2} \right|. \tag{13}$$

Since $k = 2\pi/\lambda$, one obtains for the condition of resonance

$$\frac{4b}{\lambda} = 2q + (1 + m + n). \tag{14}$$

The confocal resonator is seen to have resonances only for integer values of the quantity $4b/\lambda$. If $4b/\lambda$ is odd, $(m + n)$ must be even, likewise if $4b/\lambda$ is even, $(m + n)$ must be odd. Note that considerable degeneracy exists in the spectrum; increasing $(m + n)$ by two and decreasing q by unity gives the same frequency. The degenerate modes are orthogonal over the reflector surface since they satisfy the integral (5) with different eigenvalues. The modes have negligible axial electric and magnetic fields and thus will be designated by TEM_{mnq}, where m and n equal 0, 1, 2, $\cdots$, and refer to variations in the x and y directions, while q equals the number of half-guide wavelength variations in the z direction between reflectors.

The fractional energy loss per reflection due to diffraction effects is given by

$$\alpha_D = 1 - | \sigma_m \sigma_n |^2 = 1 - | \chi_m \chi_n |^2. \tag{15}$$

The function $1 - | \chi_m |^2$ versus c is shown in Fig. 2 for $m = 0, 1, 2$. It can be shown that Fig. 2 also gives the diffraction losses for an infinite cylindrical reflector strip of width $2a$ and radius of curvature b. The diffraction losses for various TEM_{mnq} modes are shown in Fig. 3. Note that TEM_{uvq} and TEM_{vuq} $(u \neq v)$ have the same diffraction losses; also that the diffraction losses of the TEM_{02q} and TEM_{12q} are so nearly equal that they can be plotted as one curve. As indicated previously, these last two types of modes cannot both be resonant at the same frequency. Note that the losses are primarily determined by the higher of the transverse mode numbers m, n, regardless of the field polarization.

In Fig. 3 the results of Fox and Li[6] for the plane-parallel resonator with circular reflectors are also shown. The diffraction losses for the confocal resonator are seen to be orders of magnitude *smaller* than for the plane parallel resonator. Fox and Li have also obtained numerical results for the confocal resonator with circular cross section of radius a. These are in good agreement with the results presented here, allowing for the fact that in this paper the reflectors have a square cross section of width $2a$.

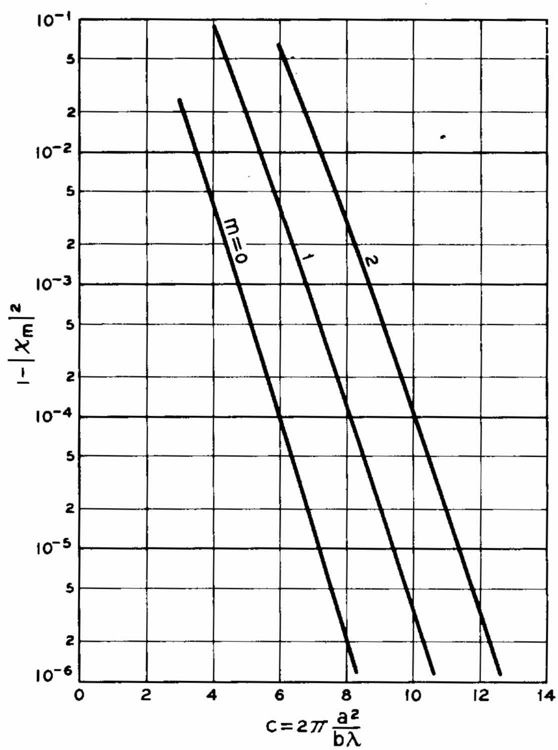

Fig. 2 — Eigenvalues of integral equation; also the diffraction losses of an infinitely long cylindrical reflector of width $2a$.

If one approximates the diffraction loss curve by a function $\alpha_D = A \times 10^{-B(a^2/b\lambda)}$, one may then show for a given reflection loss and reflector radius a that the resonator Q is a maximum as a function of the confocal spacing b when the reflection loss equals $[2.30B(a^2/b\lambda) - 1]$ times the diffraction loss. For the TEM_{00q} mode, $A = 10.9$ and $B = 4.94$; thus, if $a^2/b\lambda = 0.8$, then the diffraction loss is approximately one-eighth of the reflection loss.

The diffraction loss for the plane-parallel case assuming a uniform field and phase distribution and a diffraction angle of $\theta = \lambda/2a$ is also shown. This diffraction angle corresponds to the first Fraunhofer minimum in far field theory. For a square (or circular) reflector of side $2a$ the diffraction loss is approximately

$$\alpha_D \approx \left(\frac{a^2}{b\lambda}\right)^{-1}. \tag{16}$$

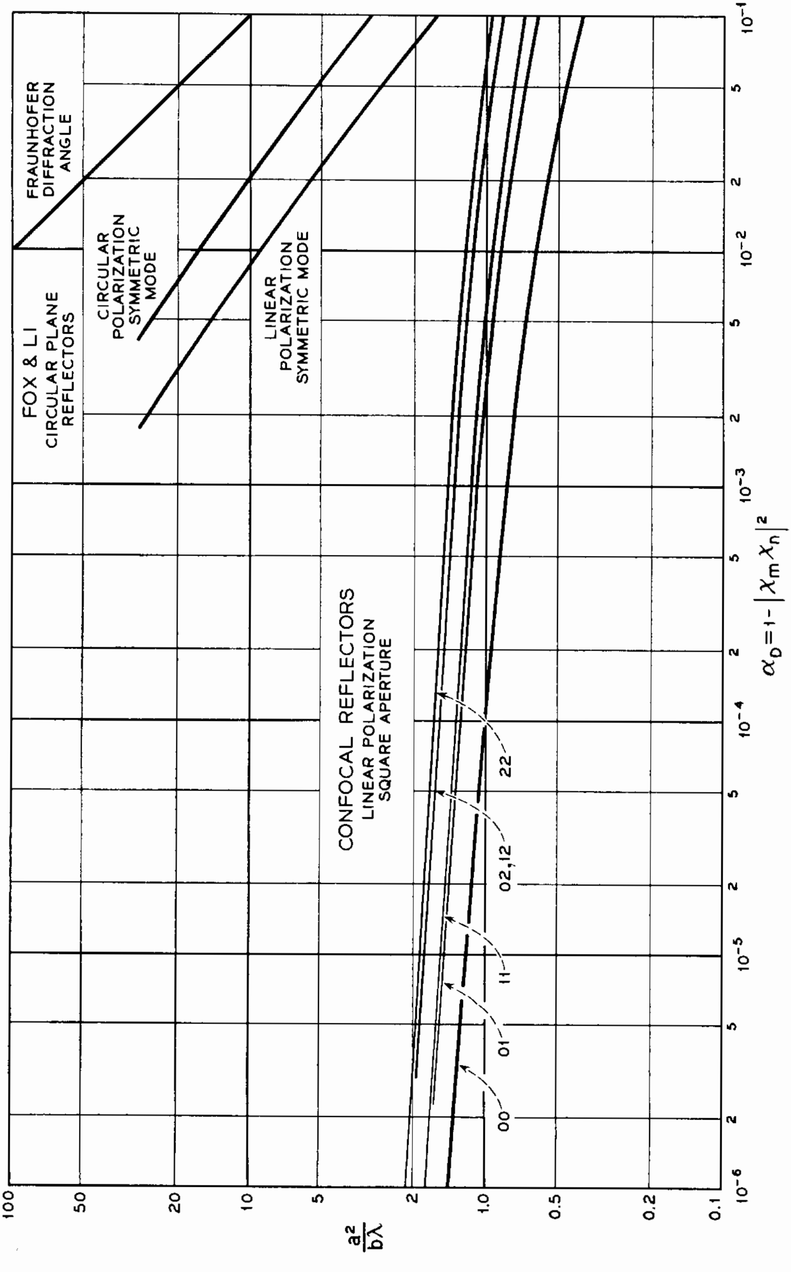

Fig. 3 — Diffraction losses for confocal and plane-parallel[6] resonators.

Fig. 3 clearly demonstrates the inadequacy of the assumption of uniform intensity distribution.

Though the eigenvalues given by (12) must be known accurately the eigenfunctions are only of approximate interest. Flammer[12] shows that, in the approximation of $\eta^2 \ll 1$ (near the center of the reflector), (10) becomes

$$F_m(X) \approx \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma(m+1)} H_m(X) e^{-\frac{1}{2}X^2} = \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma(m+1)} (-1)^m e^{+\frac{1}{2}X^2} \frac{d^m}{dX^m} e^{-X^2}. \tag{17}$$

The mode shape is thus approximately a Gaussian times a Hermite polynomial $H_m(X)$. The gamma function is arbitrarily chosen as normalization such that $F_m(X = 0) = \pm 1$ for m even:

$$\begin{aligned} F_0(c,\eta) &= e^{-\frac{1}{2}c\eta^2}, \\ F_1(c,\eta) &= \sqrt{\pi c}\,\eta\, e^{-\frac{1}{2}c\eta^2}, \\ F_2(c,\eta) &= (2c\eta^2 - 1)e^{-\frac{1}{2}c\eta^2}. \end{aligned} \tag{18}$$

The approximation involved in (17) fails away from the center of the reflector. For reasonably large values of c, however, the field is weak there, and of little interest. The diffraction losses were previously obtained from (15). Curves representing (18) for various values of c are shown in Fig. 4. The dotted curves for $c = 5$ are the true eigenfunctions $S_{0m}(c,\eta)$ as obtained from Flammer.[12]

The exponential dependence of the electric field on $c\eta^2$, which is independent of the reflector half-width a, leads one to define a "spot size" at the reflector of radius $w = w_s$, where $w^2 = x^2 + y^2$, at which the exponential term falls to e^{-1}:

$$w_s = \sqrt{\frac{b\lambda}{\pi}}. \tag{19}$$

The only effect of increasing the reflector width $2a$ is to reduce the diffraction losses; the spot size is unaffected.

If one allows the reflectors to be somewhat lossy or partially transparent, then the resonator Q is reduced over that implied by diffraction losses alone. The field distribution, i.e., the mode pattern, is *not* seriously affected so long as the losses are small and fairly uniform over the plates.

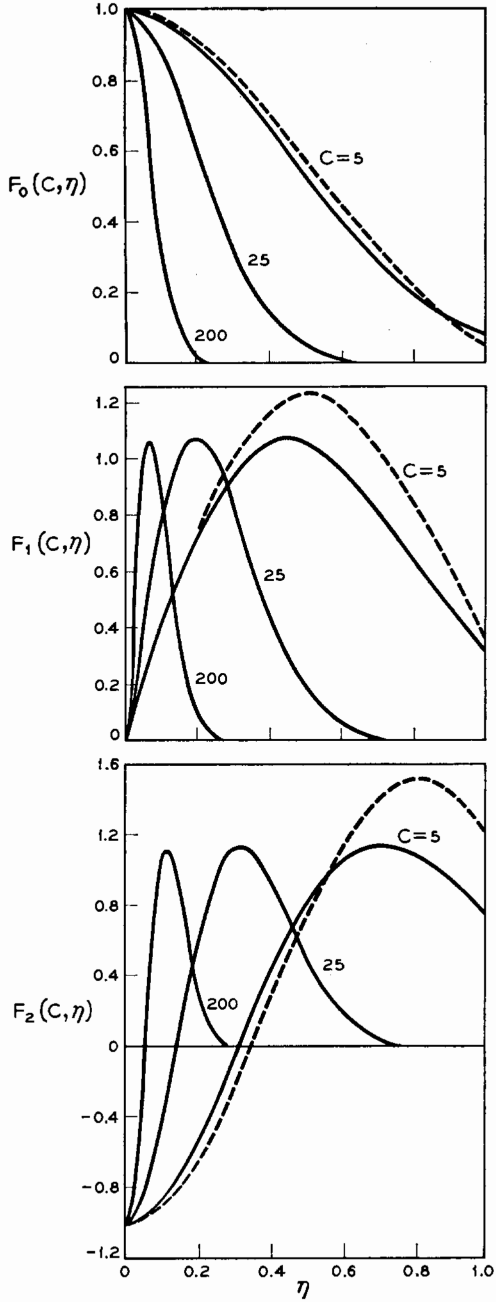

Fig. 4 — Approximate field amplitude variation versus normalized radius for various modes. The exact dependence given by the angular prolate spheroidal function $S_{0m}(c, \eta)$ is shown by dashed lines.

The electric field patterns derived thus far have all been linearly polarized. Fox and Li[6] have recognized that, by superimposing the TEM_{01q} mode linearly polarized in the x direction and the TEM_{10q} mode linearly polarized in the y direction, the lowest-order circular electric mode can result, and it has the same diffraction losses as the linearly polarized TEM_{01q} mode. Many other polarization configurations can be obtained in this manner.

IV. FIELDS OF THE CONFOCAL RESONATOR

The field over the confocal aperture has been obtained in the preceding section. The field over an arbitrary plane $z = z_0$, as in Fig. 1, is also obtainable by a straightforward application of Huygens' principle as stated in (4). The arbitrary plane z_0 may be placed outside the confocal geometry as well as inside provided one takes into account the transmission loss of the reflector. The field distribution over the confocal surface is given by $F_m(c,x/a)G_n(c,y/a)$. For large c the spheroidal functions may be approximated by the Gaussian-Hermite functions. The integral can be evaluated in the limit of $c \to \infty$.

Within these approximations, the traveling wave field of the confocal resonator resulting from the field at one of the reflectors is given by

$$\frac{E(x,y,z_0)}{E_0} = \sqrt{\frac{2}{1+\xi^2}\frac{\Gamma\left(\frac{m}{2}+1\right)\Gamma\left(\frac{n}{2}+1\right)}{\Gamma(m+1)\Gamma(n+1)}}\, H_m\left(X\sqrt{\frac{2}{1+\xi^2}}\right)$$
$$\cdot H_n\left(Y\sqrt{\frac{2}{1+\xi^2}}\right)\exp\left[-\frac{kw^2}{b(1+\xi^2)}\right] \tag{20}$$
$$\cdot \exp\left(-i\left\{k\left[\frac{b}{2}(1+\xi)+\frac{\xi}{1+\xi^2}\frac{w^2}{b}\right]-(1+m+n)\left(\frac{\pi}{2}-\varphi\right)\right\}\right),$$

where

$$w^2 = x^2 + y^2,$$
$$\xi = \frac{2z_0}{b}, \tag{21}$$
$$\tan\varphi = \frac{1-\xi}{1+\xi}.$$

When the reflecting surface is made partially transparent, as will be the case with optical or infrared masers, the field of the transmitted wave will be a traveling wave as given in (20) reduced by the transmission coefficient of the reflector. Within the resonator, the field will be a standing wave. The transverse standing wave is as given in (20) except that the exponential phase function is replaced by the sine function.

The surface of constant phase which intersects the axis at z_0 as obtained from (20) is given approximately by

$$z - z_0 \approx - \frac{\xi}{1 + \xi^2} \frac{w^2}{b}, \tag{22}$$

neglecting the small variation in φ due to variation in z. This surface is spherical, within the approximations of this paper, and has a radius of curvature b' given by

$$b' = \left| \frac{1 + \xi^2}{2\xi} \right| b. \tag{23}$$

At $\xi = \pm 1$ it coincides with the spherical reflector as expected. Also note that the symmetry or focal plane ($\xi = 0$) is a surface of constant phase.

The field distribution throughout the resonator is given by the modulus of (20). The complete field distribution within the confocal resonator is shown schematically in Fig. 5 for the low-loss TEM_{00q} mode.

The field distribution over the focal plane is less spread out than over the spherical reflectors. The field spot size over the spherical reflectors was defined by (19). In any arbitrary plane z_0 the exponential term in the field distribution falls to e^{-1} at a radius

$$w_s = \sqrt{\frac{b\lambda(1 + \xi^2)}{2\pi}}. \tag{24}$$

The smallest achievable spot size is in the focal plane at $\xi = 0$.

To obtain the radiation pattern angular beam width of the TEM_{00q} mode spherical wave, one takes the ratio of the spot diameter from (20) or (24), as $\xi \rightarrow \infty$, to the distance from the center of the resonator. The beam width between the *half-power points* is given by

$$\theta = 2 \sqrt{\frac{\ln 2}{\pi}} \sqrt{\frac{\lambda}{b}} = 0.939 \sqrt{\frac{\lambda}{b}} \text{ radians.} \tag{25}$$

V. RESONATOR WITH NONCONFOCAL SPACING

Since the surfaces of constant phase of the confocal resonator are spherical, it is apparent that (20) also represents approximately the field distribution between two spherical reflectors of arbitrary spacing. That is, any two surfaces of constant phase may be replaced by reflectors. The frequencies at which such a resonator will be resonant will of course be determined by satisfaction of the phase condition.

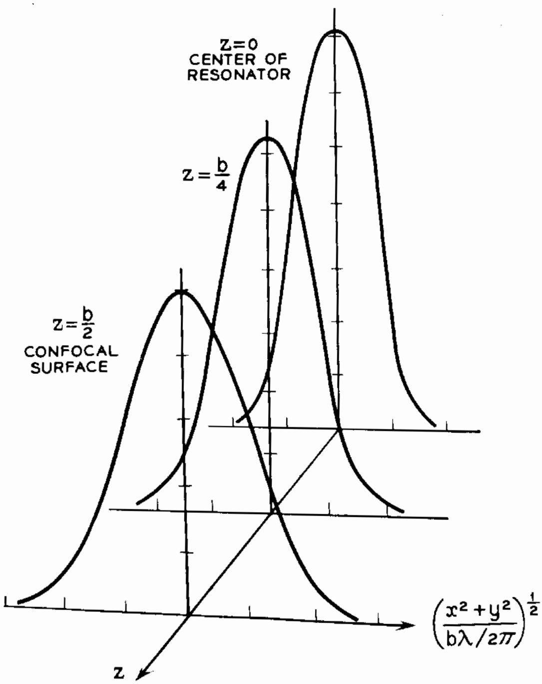

Fig. 5 — Field strength distribution within the confocal resonator for the TEM_{00q} mode.

Consider two identical spherical reflectors of radius of curvature b' spaced a distance d. The only restriction is that $b' \geqq d/2$. The confocal geometry of spacing b of which this resonator is a part is [set $\xi = d/b$ in (23)]:

$$b^2 = 2db' - d^2, \qquad b' \geqq \frac{d}{2}. \tag{26}$$

The spot size at the reflectors in the nonconfocal resonator may be immediately obtained from (24) with $\xi = \pm d/b$. It is

$$w_s' = \left(\frac{d\lambda}{\pi}\right)^{\frac{1}{2}} \left[2\frac{d}{b'} - \left(\frac{d}{b'}\right)^2\right]^{-\frac{1}{4}} \tag{27}$$

Note that the factor $[2(d/b') - (d/b')^2]$ achieves a maximum of unity, as a function of b', when $b' = d$. Thus, for a given spacing between reflectors, the spot size is a minimum for the confocal resonator.

One may estimate the loss of a nonconfocal resonator of square cross section of dimension $2a'$ on the assumption that this loss is equal to that of its equivalent confocal resonator with reflector dimensions scaled up by the ratio of their spot sizes. The equivalent confocal resonator has spacing b, and its aperture is

$$2a = 2a' \frac{w_s}{w_s'} = 2a' \left(2 - \frac{d}{b'}\right)^{\frac{1}{2}}. \tag{28}$$

The important parameter in determining losses is

$$\frac{a^2}{b\lambda} = \frac{a'^2}{d\lambda} \left[2 \frac{d}{b'} - \left(\frac{d}{b'}\right)^2\right]^{\frac{1}{2}}. \tag{29}$$

For given values of a' and d, the loss parameter is maximized, and thus losses are minimized, when $b' = d$. But this is just the confocal case. Thus the confocal geometry gives minimum spot size and minimum losses for a given spacing. If one defines the mode volume as the spot size at the reflector times the spacing, it is clear that the minimum mode volume also results from the confocal geometry. The mode volume, so defined, is

$$V = \pi w_s'^2 d = \lambda d^2 \left[2 \frac{d}{b'} - \left(\frac{d}{b'}\right)^2\right]^{-\frac{1}{2}}. \tag{30}$$

It is important to note that the results of this section are valid only when the diffraction losses derived from the "equivalent" confocal geometry are small, that is, when the reflector dimension a' is somewhat larger than the spot size. In an exact solution for the nonconfocal case one should again start from the integral (4), and clearly the field distribution and losses so derived will depart from that obtained from the equivalent confocal case if the confocal field is not substantially all intercepted by the nonconfocal reflectors. Conversely, so long as the spot size is small compared to the reflector dimension a', one expects the field distribution and losses to be very nearly correctly given by the equivalent confocal solution.

The phase shift between the two reflecting nonconfocal surfaces may be obtained from (20). The condition of resonance may then be shown to be

$$\frac{4d}{\lambda} = 2q + (1 + m + n)\left(1 - \frac{4}{\pi} \tan^{-1} \frac{b - d}{b + d}\right). \tag{31}$$

In the nonconfocal case $4d/\lambda$ is no longer necessarily an integer at resonance. It is more important, though, that the modes are no longer degenerate in $m + n$. The spectral range or mode separation for the nonconfocal resonator is given by

$$\Delta\left(\frac{1}{\lambda}\right) = \frac{1}{4d}\left[2\Delta q + \left(1 - \frac{4}{\pi}\tan^{-1}\frac{b-d}{b+d}\right)\Delta(m+n)\right]. \quad (32)$$

Note that in the confocal case the set of modes $mnq = 00q, 01q$ are maximally split in frequency, whereas if the parameter in parenthesis equals $\frac{1}{2}$ (when $d/b = 0.414$) then the $mnq = 00q, 01q, 11q, 12q$ modes are maximally split in frequency.

When $b \approx d$, (31) becomes

$$\frac{4d}{\lambda} \approx 2q + (1 + m + n)\left[1 - \frac{2}{\pi}\left(1 - \frac{d}{b}\right)\right], \quad (33)$$

where m and n are small integers and q a large integer. In the confocal case ($b = d$) note that equations (31) and (33) reduce to (14).

The theory of this section does not extend to the limit of plane-parallel reflectors, i.e., infinite radii of curvature. Let the spacing d remain fixed while b', and consequently [by (26)] the confocal radius b, approaches infinity. The spot size, as seen from (27), keeps increasing with b', and, as has been noted above, this results eventually in the breakdown of the whole idea of an equivalent confocal resonator. The relations for the nonconfocal resonator are valid as long as the reflector aperture radius a' is somewhat larger than the field spot size radius given by (27). That is, one must require

$$\frac{a'^2}{d\lambda} > \frac{1}{\pi}\left[2\frac{d}{b'} - \left(\frac{d}{b'}\right)^2\right]^{-\frac{1}{2}}. \quad (34)$$

VI. RESONANT MODES OF THE PLANE-PARALLEL RESONATOR

For comparison purposes, consider the resonances of a rectangular conducting box in the manner of Schawlow and Townes.[1] Let the dimensions be $2a \times 2a \times b$:

$$\left(\frac{2}{\lambda}\right)^2 = \left(\frac{q}{b}\right)^2 + \left(\frac{r}{2a}\right)^2 + \left(\frac{s}{2a}\right)^2, \quad (35)$$

where q, r, and s are integers. Modes where $q \gg r,s$ can be thought of physically as waves bouncing predominantly back and forth between the reflecting end plates of the rectangular box. The spectral range or mode separation is given by

$$\Delta\left(\frac{1}{\lambda}\right) = \frac{1}{2b}\left[\Delta q + \frac{1}{16}\left(\frac{b\lambda}{a^2}\right)(2r\Delta r + \Delta r^2 + 2s\Delta s + \Delta s^2)\right], \quad (36)$$

where $q \approx 2b/\lambda$ for $r, s = 1,2,3,\cdots$.

Removing the conducting side walls causes large diffraction losses for the $r = 0$ or $s = 0$ modes since they have a strong field at the edge of the reflectors. Large r or s modes represent waves traveling at a considerable angle to the normal between the reflectors and thus these modes have such large diffraction losses that they are eliminated as resolvable resonant modes. Modes with $r, s = 1,2,\cdots$, have small diffraction losses, and are approximations to the actual modes which can exist in the resonator without the conducting side walls. Fox and Li's[6] work shows that for $a^2/b\lambda$ greater than unity the mode separations of a plane-parallel Fabry-Perot are given approximately by (36), the approximation improving rapidly with increasing $a^2/b\lambda$.

The mode separation corresponding to Δr or $\Delta s = 1$ has, to the writers' knowledge, never been resolved at optical frequencies due to the large values of $a^2/b\lambda$ and low values of reflectance used. Calculations show, though, that for reflectance coefficients of about 0.99 and $a^2/b\lambda \approx 4$, such that diffraction losses are comparable with reflector losses, the resonances should be resolvable.

The mode separation due to $\Delta q = 1$ is easily resolvable and is given by $\Delta(1/\lambda) = 1/2b$. This is the spectral range as normally stated for the plane parallel Fabry-Perot interferometer. It corresponds to changing the number of half wavelengths between the reflecting surfaces by one.

The confocal resonator is resonant for integer values of $4b/\lambda$. The mode separation due to $\Delta q = 1$ is $\Delta(1/\lambda) = 1/2b$. The modes are degenerate in frequency in that for a given integer $4b/\lambda$ all TEM_{mnq} modes are resonant such that $m + n$ remains even or odd according to whether $4b/\lambda$ is odd or even. The modes of the plane-parallel Fabry-Perot are not degenerate, except for rsq and srq. A possible advantage of this degeneracy of the confocal modes will be discussed in the next section.

VII. CONFOCAL RESONATOR APPLIED TO OPTICAL MASERS

A type of solid state optical maser has recently been demonstrated by Maiman[14] and by Collins et al.[15] It consists of a fluorescent crystal material (ruby) a few centimeters in length and a few millimeters in diameter. The crystal material should be optically homogeneous. The ends of the crystal are optically flat and parallel. The ends are silver-coated for high reflectance. One of the reflecting surfaces must be slightly transparent, as the output of the optical maser is obtained through the reflecting surfaces. Thus far, silver has been used to provide the reflection, but for ultimate performance multiple-layer dielectrics[5] should be used to obtain low transmission loss as well as high reflectance. The pump power enters the fluorescent crystal from the side.

It is seen in (2) that, if diffraction losses are small compared to reflection losses, then the resonator Q is proportional to the spacing between the reflecting surfaces. Consider a confocal resonator and a plane-parallel resonator each of spacing b and of equal Q. The energy distribution in the former is more concentrated on the axis and thus the confocal resonator has a smaller effective mode volume. The volume of maser material required will thus be less for the confocal than for the plane parallel resonator. For maser oscillation the required excess density of excited states depends only on the cavity Q and in no other way upon the resonator shape.[1] The pump power is proportional to the volume of maser material times the density of excited states divided by the natural lifetime of the excited state. Thus, assuming equal Q, the confocal resonator with its smaller volume of material requires less pump power than the plane parallel resonator by the ratio of their cross-sectional areas. Snitzer[16] recently pointed out this relation between mode volume and pump power with regard to the use of optical fibers in maser applications.

The minimum volume of maser material is limited by diffraction losses. If diffraction losses are to be considerably less than 1 per cent for the lowest-order mode so as to be small compared to achievable reflection losses, then $a^2/b\lambda \gtrsim 1$. The minimum volume of maser material is then

$$V_m = \pi a^2 b \approx \pi b^2 \lambda. \tag{37}$$

If $b = 4$ cm and $\lambda = 10^{-4}$ cm, then the rod of maser material should be approximately 0.4 mm in diameter. A rod of larger diameter would waste pump power in that the field of the confocal resonator would be very weak outside this minimum diameter of material.

The analysis of the confocal resonator assumes a uniform dielectric material between the spherical reflectors. For reasons of minimizing the pump power, it is necessary to use a small diameter of maser material. Therefore, to prevent internal reflection of energy from the sides of the maser material, it may be advisable to grind rough the sides of the rod of maser material or to immerse it in a surrounding medium of equal dielectric constant. If this is not done, the energy assumed lost due to diffraction effects would not escape and the electric field pattern will not be as computed herein. A more important effect of internal reflection from the side walls would be to increase the Q of the transverse modes which would increase the spontaneous and stimulated emission power to these undesired modes, and thus increase the over-all pump power required.

The natural linewidth of the material used in an optical maser will, for reflector spacing d of a few centimeters, be large compared to the mode separation determined by integer changes in r,s for a plane-parallel resonator. Hopefully, the natural linewidth of the maser material will be less than the mode separation corresponding to integer changes in q. Thus, there is the possibility that a plane-parallel resonator optical maser may frequency wander between low-order r,s modes.

If the diffraction losses are comparable to or exceed the reflection losses for the lowest mode then, as can be seen from Fig. 3, the ratio of the Q's of the lowest two modes of the confocal resonator exceeds considerably the ratio of the Q's of the lowest two modes of the plane-parallel resonator. By the lowest order mode is meant $m = n = 0$, and $r = s = 1$, respectively, for the confocal and plane-parallel resonator. Therefore, maser oscillation is more likely to take place in *only* the lowest-order mode of the confocal than of the plane-parallel resonator. This greater loss discrimination between modes may be one of the significant advantages of the confocal resonator.

In the confocal resonator optical maser, if the maser oscillation wanders between modes the output beam pattern will change, just as in the plane-parallel resonator, but the frequency will remain fixed due to the mode degeneracy. Thus, the observed linewidth of the maser output may be narrower for the confocal resonator.

The required accuracy on the confocal condition to achieve degeneracy may be estimated from (33) and (36). It can be shown that if

$$| d/b - 1 | \approx 0.03 \qquad \text{and} \qquad (a^2/b\lambda) \approx 10,$$

the mode splitting of the near-confocal resonator equals the mode separation of the plane parallel resonator. To achieve a significantly smaller mode separation in the near-confocal resonator than the plane parallel resonator would require proportionately greater accuracy in the radius of curvature and spacing of the curved surfaces.

The plane Fabry-Perot requires accurately parallel reflecting surfaces. The confocal resonator requires only that the axis of the confocal resonator approximately coincide with the axis of the rod of maser material. The axis of the confocal resonator is the line passing through the two centers of curvature. The resonator axis must intersect the two reflecting surfaces near their center. Define the effective aperture radius as the distance from the point of intersection of the axis of the confocal resonator with the reflector surface to the nearest edge of the aperture. The diffraction losses will be approximately determined by this distance.

If the minimum diameter of maser material is used, then the axis of the confocal resonator must coincide with the material axis. Increasing the diameter of the maser material wastes pump power but relaxes the tolerance on the resonator axis.

It is well to note that a single spherical reflecting surface and a plane reflecting surface spaced by approximately half the radius of curvature will have similar properties to the confocal resonator and may be advantageous if it is desired to bring the output through a plane surface.

VIII. CONCLUSIONS

A confocal multimode resonator formed by two spherical reflectors spaced by their common radii of curvature has been considered. The mode patterns and diffraction losses have been obtained. The confocal spacing of the reflectors is found to be optimum in the sense of minimum diffraction losses and minimum mode volume.

The diffraction losses are found to be orders of magnitude smaller than those of the plane-parallel Fabry-Perot, as obtained by Fox and Li.[6] It is more important, though, that a greater diffraction loss discrimination between modes occurs, and thus oscillation in other than the lowest-order mode is less likely for the confocal resonator, assuming that diffraction losses are comparable to reflection losses.

The modes of the confocal resonator are degenerate, in that one-half of all the possible field pattern variations over the aperture are resonant at any one time. This degeneracy is split if the resonator is nonconfocal. The splitting is comparable with that of the plane-parallel resonator (with $a^2/b\lambda \approx 10$) if the spacing of the reflectors is about 3 per cent different from the common radius. The mode volume and diffraction losses are insensitive to the confocal condition.

The required volume of maser material is smaller for the confocal resonator than for the plane-parallel resonator, and thus the required pump power is less. The confocal resonator is relatively easy to adjust in that no strict parallelism is required between the reflectors. The only requirement is that the axis of the confocal resonator intersect each reflector sufficiently far from its edge so that the diffraction losses are not excessive.

The example of a confocal resonator mentioned here was taken at infrared-optical wavelengths; however, such resonators may be useful down to the millimeter wave range by virtue of their low loss. In this connection, recent work of Culshaw[17] on the plane-parallel Fabry-Perot at millimeter wavelengths is of importance.

The writers have been informed that Goubau and Schwering[18] have recently investigated diffraction losses of parabolic reflectors and that their results agree with the work presented here.

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REFERENCES

1. Schawlow, A. L. and Townes, C. H., Phys. Rev., **112**, 1958, p. 1940.
2. Prokhorov, A. M., J.E.T.P., **34**, 1958, p. 1658.
3. Dicke, R. H., U.S. Patent 2,851,652, September 9, 1958.
4. Meissner, K. W., J. Opt. Soc. Am., **31**, 1941, p. 405; **32**, 1942, p. 185.
5. Heavens, O. S., *Optical Properties of Thin Films*, Butterworths, London, 1955.
6. Fox, A. G. and Li, T., this issue, p. 453; Proc. I.R.E., **48**, 1960, p. 1904.
7. Lewis, W. D., private communication.
8. Connes, P., Revue d'Optique, **35**, 1956, p. 37; J. Phys. Radium, **19**, 1958, p. 262.
9. Jenkins, F. A. and White, H. E., *Fundamentals of Optics*, 3rd ed., McGraw-Hill, New York, 1957.
10. Silver, S., *Microwave Antenna Theory and Design*, M.I.T. Radiation Laboratory Series, Vol. 12, McGraw-Hill, New York, 1949.
11. Slepian, D. and Pollak, H. O., B.S.T.J., **40**, 1961, p. 43.
12. Flammer, C., *Spheroidal Wave Functions*, Stanford Univ. Press, Palo Alto, Calif., 1957.
13. Slepian, D. and Pollak, H. O., private communication.
14. Maiman, T. H., Nature, **187**, 1960, p. 493.
15. Collins, R. J., Nelson, D. F., Schawlow, A. L., Bond, W., Garrett, C. G. B. and Kaiser, W., Phys. Rev. Letters, **5**, 1960, p. 303.
16. Snitzer, E., J. Appl. Phys., **32**, 1961, p. 36.
17. Culshaw, W., I.R.E. Trans., **MTT-7**, 1959, p. 221; **MTT-8**, 1960, p. 182.
18. Goubau, G. and Schwering, F., U.R.S.I.-I.R.E. Spring Meeting, Washington, May 1960.