Fifth-order corrected electromagnetic field components for a fundamental Gaussian beam

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Fifth-order corrected expressions for the electromagnetic field components of a monochromatic fundamental Gaussian beam (i.e., a focused TEM_{00} mode laser beam) propagating within a homogeneous dielectric media are derived and presented. Calculations of relative error indicate that the fifth-order Gaussian beam description provides a significantly improved solution to Maxwell's equations in comparison with commonly used paraxial (zeroth-order) and first-order Gaussian beam descriptions.

I. INTRODUCTION

In two recent papers,[1,2] theoretical calculations have been presented investigating the electromagnetic interaction of a monochromatic fundamental Gaussian beam (i.e., a TEM_{00} mode focused laser beam) with a homogeneous spherical particle. A mathematical description of the electromagnetic field components of the incident Gaussian beam which accurately satisfies Maxwell's equations is required for these calculations. In Refs. 1 and 2, first-order corrected expressions developed by Davis[3] were used for the incident Gaussian beam description. This first-order corrected Gaussian beam description, which is consistent with the first-order corrected expressions of Lax, Louisell, and McKnight[4] and the development of Simon, Sudarshan, and Muhunda,[5] has been found to give good results when the beam waist radius (w_0) is much greater than the wavelength (λ).

For tightly focused beams, where the beam waist radius is of the same order as the wavelength, the first-order corrected Gaussian beam description satisfies Maxwell's equations less accurately and the electromagnetic field calculations using the spherical particle/arbitrary beam interaction theory of Ref. 1 likewise become less accurate. In Ref. 3, Davis presents a procedure for developing a higher-order corrected Gaussian beam description and provides expressions for two of the three electric field components to third order in the parameter s, where

$$s = \frac{1}{kw_0} = \frac{1}{2\pi}\left(\frac{\lambda}{w_0}\right). \tag{1}$$

However, Davis's expressions are for vacuum conditions only, do not include the third electric field component or the magnetic field components, and, beyond first order, lack symmetry with regard to the electric and magnetic field components.

In this paper, a modification and extension of the procedure introduced by Davis[3] is used to derive *fifth-order* corrected expressions for all six electromagnetic field components. These expressions are symmetrical with regard to the electric and magnetic field components and are applicable for dielectric media. A comparison of relative error as a function of order and tightness of focus is also presented. These results should be of interest to anyone requiring an accurate Gaussian beam description, particularly for tightly focused conditions.

II. THEORETICAL DEVELOPMENT

The propagation of a monochromatic beam within isotropic, homogeneous, nonmagnetic ($\mu = 1$), nonconducting ($\sigma = 0$) dielectric media is considered. A harmonic time dependence of $e^{+i\omega t}$ is assumed. (The $e^{+i\omega t}$ is dropped from all subsequent time-dependent terms.) For these conditions, Maxwell's equations can be written in the following form:

$$\nabla\cdot\mathbf{E} = 0, \tag{2}$$

$$\nabla\times(\mathbf{H}/\sqrt{\epsilon}) - ik\,\mathbf{E} = 0, \tag{3}$$

$$\nabla\times\mathbf{E} + ik(\mathbf{H}/\sqrt{\epsilon}) = 0, \tag{4}$$

and

$$\nabla\cdot(\mathbf{H}/\sqrt{\epsilon}) = 0, \tag{5}$$

where $k = \sqrt{\epsilon}\omega/c = 2\pi/\lambda$ is the wave number within the media. In the Lorentz gauge, a vector potential $\mathbf{A}$ can be defined such that if $\mathbf{A}$ is a solution of the Helmholtz equation,

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = 0, \tag{6}$$

then

$$\mathbf{E} = -(i/k)\nabla(\nabla\cdot\mathbf{A}) - ik\,\mathbf{A} \tag{7}$$

and

$$\mathbf{H}/\sqrt{\epsilon} = \nabla\times\mathbf{A} \tag{8}$$

satisfy Maxwell's equations, Eqs. (2)–(5).

Davis[3] observed that, for a beam propagating in the $+z$ axis direction, if the vector potential is assumed linearly polarized in the x axis direction,

$$\mathbf{A} = A\hat{x} \tag{9}$$

where

$$A = \psi(x,y,z)e^{-ikz}, \tag{10}$$

then Eqs. (6)–(8) provide a linearly polarized (predominate electric field polarization in the x axis direction) Gaussian beam. Foreseeing a solution similar to the well-established paraxial (zeroth-order) Gaussian beam solution (see, for example, Yariv[6]), the spatial coordinates transverse to the direction of propagation are normalized relative to the beam waist radius, $\xi = x/w_0$ and $\eta = y/w_0$, and the spatial coordinate in the direction of propagation is normalized relative to the diffraction length, $\zeta = z/(kw_0^2)$. With this change of spatial variables, the expression for the vector potential becomes

$$A = \psi(\xi,\eta,\zeta)e^{-i\zeta/s^2} \tag{11}$$

and Helmholtz equation, in terms of ψ, can be rearranged as

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} - 2i\frac{\partial}{\partial\zeta}\right)\psi = -s^2\frac{\partial^2\psi}{\partial\zeta^2}. \tag{12}$$

If the parameter s, defined in Eq. (1), is assumed small ($w_0 \gg \lambda$) then from Eq. (12) it is apparent that ψ can be expanded as a sum of even powers of s so that

$$\psi = \psi_0 + s^2\psi_2 + s^4\psi_4 + \cdots, \tag{13}$$

where

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} - 2i\frac{\partial}{\partial\zeta}\right)\psi_0 = 0, \tag{14}$$

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} - 2i\frac{\partial}{\partial\zeta}\right)\psi_2 = -\frac{\partial^2\psi_0}{\partial\zeta^2}, \tag{15}$$

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} - 2i\frac{\partial}{\partial\zeta}\right)\psi_4 = -\frac{\partial^2\psi_2}{\partial\zeta^2}, \tag{16}$$

etc.

Equation (14) is the familiar paraxial beam equation,[6] which has the solution

$$\psi_0 = iQ\exp(-i\rho^2 Q), \tag{17}$$

where $\rho^2 = \xi^2 + \eta^2$ and $Q = 1/(i + 2\zeta)$. Substituting Eq. (17) into Eq. (15), Davis found

$$\psi_2 = (2iQ + i\rho^4 Q^3)\psi_0. \tag{18}$$

Continuing and substituting Eq. (18) into Eq. (16) it follows that

$$\psi_4 = (-6Q^2 - 3\rho^4 Q^4 - 2i\rho^6 Q^5 - 0.5\rho^8 Q^6)\psi_0. \tag{19}$$

Using

$$\mathbf{A} \approx (\psi_0 + s^2\psi_2 + s^4\psi_4)e^{-i\zeta/s^2}\hat{x} \tag{20}$$

with Eqs. (7) and (8) would give expressions for electromagnetic field components to fifth order in s. However, beyond first order, these expressions for the electric and magnetic field components lack symmetry. [Note since $\mathbf{A} = A\hat{x}$, Eq. (8) indicates $H_x = 0$ for all orders of s.] In order to develop a Gaussian beam description for which the electric and magnetic field component expression have symmetry, the derivation was repeated with

$$\mathbf{H}'/\sqrt{\epsilon} = +(i/k)\nabla(\nabla\cdot\mathbf{A}') + ik\,\mathbf{A}' \tag{21}$$

and

$$\mathbf{E}' = \nabla\times\mathbf{A}', \tag{22}$$

where

$$\nabla^2\mathbf{A}' + k^2\mathbf{A}' = 0 \tag{23}$$

and

$$\mathbf{A}' = A'\hat{y} = \psi'(x,y,z)e^{-ikz}\hat{y}. \tag{24}$$

The solution of Eqs. (7) and (8) was appropriately added to the solution of Eqs. (21) and (22) and the result divided by two. The resulting electromagnetic field component expressions, to fifth order in the parameter s, are as follows:

$$\begin{aligned} E_x = E_0\{1 &+ s^2(-\rho^2Q^2 + i\rho^4Q^3 - 2Q^2\xi^2) \\ &+ s^4[+2\rho^4Q^4 - 3i\rho^6Q^5 - 0.5\rho^8Q^6 \\ &+ (8\rho^2Q^4 - i2\rho^4Q^5)\xi^2]\}\psi_0 e^{-i\zeta/s^2}, \end{aligned} \tag{25}$$

$$E_y = E_0\{s^2(-2Q^2\xi\eta) + s^4[(8\rho^2Q^4 - 2i\rho^4Q^5)\xi\eta]\}\psi_0 \times e^{-i\zeta/s^2}, \tag{26}$$

$$\begin{aligned} E_z = E_0\{&s(-2Q\xi) + s^3[(+6\rho^2Q^3 - 2i\rho^4Q^4)\xi] \\ &+ s^5[(-20\rho^4Q^5 + 10i\rho^6Q^6 + \rho^8Q^7)\xi]\}\psi_0 e^{-i\zeta/s^2}, \end{aligned} \tag{27}$$

$$H_x = \sqrt{\epsilon}E_0\{s^2(-2Q^2\xi\eta) + s^4[(8\rho^2Q^4 - 2i\rho^4Q^5)\xi\eta]\}\psi_0 \times e^{-i\zeta/s^2}, \tag{28}$$

$$\begin{aligned} H_y = \sqrt{\epsilon}E_0\{1 &+ s^2(-\rho^2Q^2 + i\rho^4Q^3 - 2Q^2\eta^2) \\ &+ s^4[+2\rho^4Q^4 - 3i\rho^6Q^5 - 0.5\rho^8Q^6 \\ &+ (8\rho^2Q^4 - i2\rho^4Q^5)\eta^2]\}\psi_0 e^{-i\zeta/s^2}, \end{aligned} \tag{29}$$

and

$$\begin{aligned} H_z = \sqrt{\epsilon}E_0\{&s(-2Q\eta) + s^3[(+6\rho^2Q^3 - 2i\rho^4Q^4)\eta] \\ &+ s^5[(-20\rho^4Q^5 + 10i\rho^6Q^6 + \rho^8Q^7)\eta]\}\psi_0 e^{-i\zeta/s^2}, \end{aligned} \tag{30}$$

where ψ_0 is given by Eq. (17).

In Eqs. (25)–(30), E_0 is the electric field amplitude at the focal point of the beam ($\xi = \eta = \zeta = 0$) which can be related to the beam power P by

$$|E_0|^2 = \frac{16P}{\sqrt{\epsilon}cw_0^2(1 + s^2 + 1.5s^4)}. \tag{31}$$

To zeroth order in s, Eqs. (25)–(30) provide the familiar paraxial Gaussian beam description.[6] To first order in s, Eqs. (25)–(30) are identical to the first-order corrected expressions of Davis[3] and Lax, Louisell, and McKnight[4] and the development of Simon, Sudarshan, and Mukunda.[5] Equations (25)–(30) are for a harmonic time dependence of $e^{+i\omega t}$. If a harmonic time dependence of $e^{-i\omega t}$ is assumed, then the complex conjugate of Eqs. (25)–(30) can be used.

III. RELATIVE ACCURACY

The fifth-order corrected Gaussian beam description was verified by directly substituting the electromagnetic field components given by Eqs. (25)–(30) into Maxwell's equations, Eqs. (2)–(5), and calculating the relative percent error. [The relative percent error was computed by taking the percent deviation of the magnitude of the left-hand side of each equation from zero relative to $k|\mathbf{E}|$ for Eqs. (2) and (3) and relative to $k|\mathbf{H}|/\sqrt{\epsilon}$ for Eqs. (4) and (5).] Table I provides a comparison of the percent error for $s = 0.02, 0.05, 0.10, 0.20, 0.30$, and 0.40 for zeroth- to fifth-order Gaussian beam descriptions. Both the average percent error and the maximum percent error were calculated for 216 spatial positions surrounding the focal point consisting of all combinations of $\xi, \eta, \zeta = 0.0, 0.1, 0.2, 0.5, 1.0$, and 1.5. As shown in

TABLE I. Average percent error and maximum percent error of solution to Maxwell's equations for zeroth- to fifth-order Gaussian beam descriptions vs s. Percent error calculated for 216 points consisting of all combinations of ξ, η, $\zeta = 0.0$, 0.1, 0.2, 0.5, 1.0, and 1.5.

	$s =$	0.02	0.05	0.10	0.20	0.30	0.40
s^0	avg%	0.817	2.10	4.37	9.47	15.3	21.8
	max%	3.07	7.94	16.8	37.0	60.8	88.0
s^1	avg%	1.73×10^{-2}	0.111	0.457	1.90	4.33	7.74
	max%	9.28×10^{-2}	0.603	2.51	10.3	22.6	38.4
s^2	avg%	6.43×10^{-4}	1.05×10^{-2}	8.85×10^{-2}	0.757	2.56	5.89
	max%	8.23×10^{-3}	0.133	1.14	10.7	31.9	49.3
s^3	avg%	2.36×10^{-5}	9.58×10^{-4}	1.61×10^{-2}	0.277	1.44	4.25
	max%	1.97×10^{-4}	8.26×10^{-3}	0.144	2.51	19.1	36.0
s^4	avg%	1.15×10^{-6}	1.19×10^{-4}	4.10×10^{-3}	0.148	1.13	3.85
	max%	2.46×10^{-5}	2.52×10^{-3}	8.85×10^{-2}	3.99	38.2	54.0
s^5	avg%	5.13×10^{-8}	1.27×10^{-5}	8.69×10^{-4}	6.19×10^{-2}	0.725	3.34
	max%	7.58×10^{-7}	1.99×10^{-4}	1.40×10^{-2}	1.19	22.2	36.6

Table I, the fifth-order Gaussian beam description gives a significant improvement in accuracy in comparison with the commonly used zeroth- and first-order descriptions. As might be expected, the fifth-order Gaussian beam expressions become less accurate as s approaches one. According to these calculations, if a deviation of 1% is acceptable, then the fifth-order description can be used for s less than about 0.2. An s value of 0.2 corresponds to a beam waist radius to wavelength ratio of about 0.8.

IV. DISCUSSION

The fifth-order Gaussian beam description has been used with the spherical particle/arbitrary beam interaction theory of Ref. 1 with improved results in comparison with the previously used first-order Gaussian beam description (as evidenced by improved matching of boundary conditions across the surface of the sphere, etc.).

The procedure outlined here could be continued to develop higher-order corrected Gaussian beam descriptions. In addition, as suggested by Zauderer,[7] the transverse spatial derivatives of Eqs. (25)–(30) (and linear combinations of these derivatives) perhaps could be used to construct corrected expressions for higher-mode Gaussian beams. This advanced work is presently under consideration.

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