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The Poynting vector in Laguerre–Gaussian beams and the interpretation of their angular momentum density

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Abstract

We use the local linear momentum density in Laguerre–Gaussian beams to investigate classically the trajectory of the Poynting vector for light of any polarisation. Some of the difficulties of interpretation of the local angular momentum density are discussed. © 2000 Published by Elsevier Science B.V.

Keywords: Poynting vector; Angular momentum density; Polarised light

1. Introduction

The rate of electromagnetic energy flow per unit area, the Poynting vector, is defined and discussed in all books on electromagnetism (see, e.g., Ref. [1]). The vector and its associated linear momentum density is, however, invariably examined for plane waves or at least waves with a plane wavefront. Recent interest in orbital angular momentum of light beams [2] has led to a considerable literature relating to Laguerre–Gaussian beams which have helical wavefronts. The angular momentum density, $\boldsymbol{j}$, is related [1] to the linear momentum density, $\boldsymbol{p}$, through

$$\boldsymbol{j} = \boldsymbol{r} \times \boldsymbol{p}, \tag{1}$$

where

$$\boldsymbol{p} = \varepsilon_0 \boldsymbol{E} \times \mathbf{B}, \tag{2}$$

and the total angular momentum is naturally derived from the linear momentum [3]. The emphasis in the literature is, for obvious reasons, on orbital angular momentum, but as light of arbitrary polarisation is considered, it also accounts for spin angular momentum. In this work, we use the components of the linear momentum density to determine the behaviour of the Poynting vector for Laguerre–Gaussian modes and then examine the meaning of the angular momentum density.

2. Linear momentum density

It is easy to show within the paraxial approximation [2,3] that for any mode with an amplitude given by

$$u(r,\phi,z) = u(r,z)\exp(\mathrm{i}l\phi), \tag{3}$$

the r, φ and z components of linear momentum density, $\varepsilon_0 \boldsymbol{E} \times \boldsymbol{B}$, for a circularly polarised beam propagating in the z-direction are given by

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$$p_r = \varepsilon_0 \frac{\omega k r z}{(z_R^2 + z^2)} |u|^2,$$
$$p_\phi = \varepsilon_0 \left[\frac{\omega l}{r} |u|^2 - \frac{1}{2} \omega \sigma \frac{\partial |u|^2}{\partial r} \right],$$
$$p_z = \varepsilon_0 \omega k |u|^2. \qquad (4)$$

The component p_r, relates to the spread of the beam as it propagates. The first term of the azimuthal component p_ϕ depends on l, where $l\hbar$ has been identified as the orbital angular momentum per photon [3]. Its second term relates to the spin contribution, where $\sigma = \pm 1$ for left-handed and right-handed circularly polarised light, respectively, and $-1 < \sigma < 1$ describes elliptically polarised light. It is this contribution which leads to the spin angular momentum of the beam. The final component, p_z, is the linear momentum in the direction of propagation.

The relative size of the components determines the trajectory of the Poynting vector (Fig. 1). We have investigated elsewhere this trajectory with respect to the orbital contribution alone [4]. This

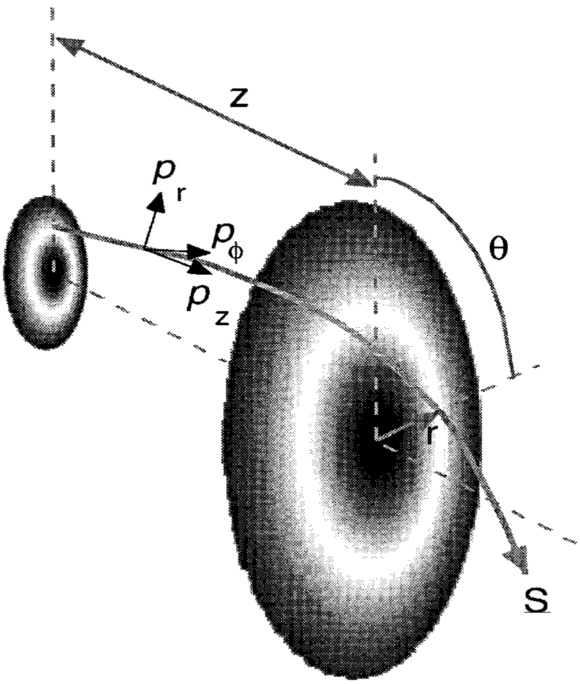

Fig. 1. The trajectory of the Poynting vector and the components of linear momentum density.

was for linearly polarised light where $\sigma = 0$; most of the interesting terms investigated here were therefore zero and played no role. When both orbital and spin momenta are included, the rate of change of azimuthal angle of the trajectory of the Poynting vector with z is given by

$$\frac{\partial \theta}{\partial z} = \frac{1}{r} \frac{p_\phi}{p_z} = \frac{l}{kr^2} - \frac{\sigma}{2kr} \frac{1}{|u|^2} \frac{\partial |u|^2}{\partial r}. \qquad (5)$$

A Laguerre–Gaussian mode has an amplitude of

$$u_{pl}^{\mathrm{LG}} = \frac{C_{pl}^{\mathrm{LG}}}{w(z)} \left(\frac{r\sqrt{2}}{w(z)} \right)^{|l|} \exp\left[-\frac{r^2}{w^2(z)} \right] L_p^{|l|} \times \exp\left(-\frac{\mathrm{i}kr^2 z}{2(z^2 + z_R^2)} \right) \exp(-\mathrm{i}l\phi) \times \exp\left(\mathrm{i}(2p + |l| + 1) \tan^{-1} \frac{z}{z_R} \right), \qquad (6)$$

where C_{pl}^{LG} is the normalisation constant; $L_p^{|l|}(2r^2/w^2(z))$ is a generalised Laguerre polynomial; the radius of the beam at position z is $w(z)$ where

$$w(z)^2 = \frac{2}{k} \frac{z_R^2 + z^2}{z_R}, \quad \text{and} \quad (2p + |l| + 1) \tan^{-1} \frac{z}{z_R}$$

is the Gouy phase, and z_R, the Rayleigh range. Here, p and l are mode indices, where l is the number of intertwined helices, and p, the number of radial nodes. It follows that for such a mode, Eq. (5) becomes

$$\frac{\partial \theta}{\partial z} = \frac{l}{kr^2} - \frac{\sigma |l|}{kr^2} + \frac{2\sigma}{kw^2(z)} + \frac{4\sigma}{kw^2} \frac{L_{p-1}^{|l+1|}(2r^2/w^2(z))}{L_p^{|l|}(2r^2/w^2(z))}, \qquad (7)$$

where k may be expressed in terms of $w(z)$, as above.

For modes with $p = 0$, the final term in Eq. (7) is always zero. When $l = 0$ and $p = 0$ and the Laguerre–Gaussian is identical to the Hermite Gaussian (0,0)-mode, the orbital angular momentum is zero and the rotation of the Poynting vector for a polarised beam arises from the effect of spin alone. The first two terms are also zero and $\theta = \sigma \arctan z/z_R$ and $\theta \to \pi/2$ as $z \to \infty$.

For a single ringed Laguerre–Gaussian mode when $p = 0$ but $l \neq 0$,

$$\frac{\partial\theta}{\partial z} = \frac{l}{kr^2}\left(1 - \frac{\sigma|l|}{l}\right) + \frac{2\sigma}{kw^2}. \tag{8}$$

At peak intensity when $r^2 = w^2(z)|l|/2$, we find for all values of l and σ that $\theta = \pm\arctan z/z_{\mathrm{R}}$. The trajectory of the Poynting vector is in general a spiral with a pitch that depends on the radius. Surprisingly, at peak intensity, the pitch is such that it becomes a straight line skewed with respect to the beam axis. Such a trajectory is consistent with a ray optical model where orbital angular momentum may be represented by skew rays in the optical beam.

When $\sigma = \pm 1$, that is for circularly polarised light, we see that if the handedness of spin and orbit are the same, then $\theta = \arctan z/z_{\mathrm{R}}$ for all l and for all positions across the beam amplitude distribution, not just at the peak intensity.

In general, for a mode of arbitrary index p and any value of l and σ,

$$\theta = \left\{ \frac{l - \sigma\frac{|l|}{l}}{2}\frac{w^2(0)}{r^2(0)} + \sigma + 2\sigma\frac{L_{p-1}^{|l+1|}(2r^2/w^2(z))}{L_p^{|l|}(2r^2/w^2(z))} \right\} \times \arctan\frac{z}{z_{\mathrm{R}}}. \tag{9}$$

All previous values may be found from this, as well as the solely spin-dependent contribution which arises for higher-order modes. As any arbitrary light beam can be expressed in terms of a set of Laguerre–Gaussian modes, this result is of general applicability.

3. Angular momentum density

We now consider the angular momentum density, which is closely related to the azimuthal component of the Poynting vector. It follows from the linear momentum densities already given in Eq. (4), that the angular momentum density in the direction of propagation z derived from $\boldsymbol{j} = \boldsymbol{r} \times \boldsymbol{p}$ is

$$j_z = \varepsilon_0\left\{\omega l|u|^2 - \frac{1}{2}\omega\sigma r\frac{\partial|u|^2}{\partial r}\right\}. \tag{10}$$

When this quantity is integrated over the beam and divided by the energy, we find [3]

$$\frac{J_z}{W} = \frac{\int j_z r\,\mathrm{d}r\,\mathrm{d}\phi}{\int w_z r\,\mathrm{d}r\,\mathrm{d}\phi} = \frac{l\int|u|^2 r\,\mathrm{d}r\,\mathrm{d}\phi}{\omega\int|u|^2 r\,\mathrm{d}r\,\mathrm{d}\phi} - \frac{\sigma}{2\omega}\frac{\int r^2\frac{\partial|u|^2}{\partial r}\mathrm{d}\phi}{\int|u|^2 r\,\mathrm{d}r\,\mathrm{d}\phi}. \tag{11}$$

When the integration is over the whole cross-section of the beam, the total angular momentum J_z divided by total energy W reduces simply to

$$\frac{J_z}{W} = \frac{l}{\omega} + \frac{\sigma}{\omega}. \tag{12}$$

When the integration extends over only part of the beam, the first term is always constant, l/ω, while the second term depends on the intensity gradient.

If the energy is radiated in photons of energy $\hbar\omega$, then the angular momentum is transferred in units of $\hbar$. As Biedenharn and Louck [5], "It is in this way that Maxwell's equations, which do not contain Planck's constant, are nonetheless compatible with the quantum mechanics of photons. (The agreement fails for order $\hbar^2$.)" It is this simple relationship which has led to the quantity $l\hbar$ being designated as the orbital angular momentum [3]. There are, however, problems. A classical treatment is not straightforwardly in accord with the quantum theory. For example, the square of the angular momentum divided by the square of the energy for a classical electromagnetic multipole (J, J_z) turns out to be J_z^2/ω^2 and not, as might have been expected, $J(J+1)/\omega^2$. In fact, Morette-de Witt and Jensen [6] showed using quantised fields that the correct value is $\{N^2J_z^2 + N[J(J+1) - J_z^2]\}/N^2\omega^2$, where N is the number of quanta in the mode (J, J_z). For large N, the classical value J_z^2/ω^2 occurs, but at the single photon level, we get $J(J+1)/\omega^2$.

A more serious problem is one which has been tackled ever since Beth's famous experimental determination of $\hbar$ [7]. A circularly polarised plane wave with electric and magnetic fields only in the x- and y-directions has a linear momentum density only in the z-direction. When this is crossed with r to give the angular momentum density, there is no contribution in the z-direction. Thus, such a beam has no angular momentum to transfer to a waveplate, yet, Beth was able to make such a transfer – a paradox.

The explanation has a considerable literature (see list of references spanning 1936–54 given in the book by Jauch and Rohrlich [8]). Probably Simmons and Guttmann [9] give the most readily accessible discussion of the problem in their unusual, but fascinating, book. They do not solve the wave equation directly but their rejection of second-order derivatives when examining Maxwell's equations, makes their argument equivalent to the paraxial approximation. The explanation of the paradox is that the finite extent of the waveplate, or the finite extent of the field whichever is smaller, creates an aperturing that prevents the field being a plane wave as its extent is no longer infinite. It then follows that in the direction of propagation, there are electric and magnetic field components, which give azimuthal and radial components to the momentum density. When these are crossed with r, there is an angular momentum density in the z-direction. The essential part of the Simmons and Guttmann's argument is that the aperturing at the edge of the waveplate creates a large intensity gradient, and in this region, the angular momentum components are, therefore, very strong. The term involving the intensity gradient shown in Eq. (10) is to be found in the book in the integrands of the two unnumbered equations prior to Eq. (9.22). Remarkably, when averaged with the zero angular momentum over the area of the waveplate, the result is that the spin angular momentum divided by energy is σ/ω per photon, as expected.

The problem of Gaussian beams is, however, not quite the same. They already have a natural radial intensity gradient because they go to zero intensity at infinity. Such fields, irrespective of the presence of any waveplate, have field components in the direction of propagation. It is because these fields exist that there is a permanent spin angular momentum density, and the fundamental paradox associated with the idealised plane wave is absent. For Laguerre–Gaussian beams, the fields are more complex because of the helical wavefronts and their associated azimuthal phase term. Indeed, it is these more complex axial fields which specifically create the orbital angular momentum with which L–G modes are now associated. The key question is whether the edge effect of a waveplate, or some other aperture or obstacle, still plays as significant a role as that played in the case of a plane wave? The question is important because, as the spin angular momentum density depends on intensity gradient and because these fields do not have uniform intensity, a different amount of angular momentum might be expected to be transferred at different positions in the wavefront.

For the commonly occurring $p = 0$ modes, the expression for total angular momentum divided by total energy Eq. (11) is readily shown to be

$$\frac{J_z}{W} = \frac{l}{\omega} - \frac{\sigma}{\omega} \times \frac{\int_0^b \frac{1}{2} r^2 \left\{ \frac{2|l|}{r} - \frac{4r}{w^2(z)} \right\} r^{2|l|} \exp\left[-\frac{2r^2}{w^2(z)} \right] \mathrm{d}r}{\int_0^b r\, r^{2|l|} \exp\left[-\frac{2r^2}{w^2(z)} \right] \mathrm{d}r}. \quad (13)$$

The local value of the orbital angular momentum given by the first term is of little interest to this discussion, as it is a constant. We see that the second term although associated with spin also depends on the magnitude of l. This does not imply a coupling between spin and orbital angular momentum, it is merely a reflection of the fact that the spin contribution depends on the field intensity gradient and this depends on $|l|$. The integral is easily evaluated to show that when the beam is apertured at any radius b to give a rapid decline in intensity to zero, we find the result given in expression (12) in just the same way as for a plane wave. If the expression equivalent to Eq. (13) is derived for modes with a value of $p > 0$, then it may be readily shown that the parts of the wavefront that lie between any of the regions of zero intensity also integrate neatly to the same result; there is, of course, no plane wave equivalent.

What, however, remains of interest is the possibility that there is some regime where there might be a difference in the detailed behaviour of the spin angular momentum of Laguerre–Gaussian beams compared with plane waves. The intensity-gradient-dependent local value of spin is given by the second term of Eq. (10). If we write the local spin angular momentum per energy, we find

$$\frac{j_{z,\mathrm{spin}}}{w} = -\frac{\sigma r}{2\omega} \frac{\left(\partial |u|^2 / \partial r\right)}{|u|^2} \tag{14}$$

which for a $p = 0$ mode, gives a local spin angular momentum per photon of

$$j_{z,\mathrm{spin}} = -\sigma\hbar\left(|l| - \frac{2r^2}{w^2(z)}\right). \tag{15}$$

Even though the beam has been macroscopically polarised with a unique value of σ, the z-component of spin per photon in a $(p = 0, l)$-mode clearly depends in this picture on the position across the beam; in some places, it is negative and in others it is positive, and its absolute value is unlimited. It is easy to see that there is a spread of values of $j_{z,\mathrm{spin}}$ and that it is not limited to a maximum value of unity. The spin angular momentum goes from $-|l|\sigma$ near to the beam axis, to zero at the peak intensity and thereafter becomes positive and arbitrarily high. Nevertheless, on average, the spin per photon remains $\sigma\hbar$.

Is it really possible that matter can respond to the local value of angular momentum, which may not only be different in magnitude but have a different sign from that expected by the state of polarisation of the beam? Clearly any attempt to separate the different regions by means of an aperture is doomed to failure, because the rapid intensity variation of the edge effect will again produce the expected, "correct" result. It would seem that an experiment involving atoms might be more promising. If there were still an equivalent to the edge effect, the dipole approximation would seem to have no meaning, and yet, it is the basic assumption of virtually all theoretical approaches to atom–field interactions.

Another concern is that it appears that a net spin component per photon greater than unity is possible; it is hard to see what this really means. Nevertheless, in the Simmons and Guttmann exposition for a plane wave at the edge of the waveplate, the same thing is true, otherwise the net angular momentum per unit energy cannot average to unity, as in some areas the density is zero. But, as they write, "a classical quantity associated with the electromagnetic field does not necessarily indicate the value of that quantity which will be measured." There is considerable scope for the classical argument to fail; it could well be that there are significantly different observable effects at the single photon level. Nevertheless, the classical approach although simple, has been remarkably successful so far [2] in accounting for so many phenomena. It would be very exciting to find a regime where the experimental results manifest the sign and the varying value of the local angular momentum density in the way the above discussion suggests.

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