straight line which if extended passed through the origin. It would thus seem that, neglecting surface tensional effects, the oil should show no coiling effect at zero height of fall, but should begin to coil slowly as soon as the orifice is no longer in contact with the plate.

Slight fluctuations in the stream diameter caused by air currents and inconsistencies in the oil itself introduced small deviations in the values of X as measured with the traveling microscope. In the greater-height-of-fall region where X was determined from pictures, small variations in caliper measurements yielded a large percentage deviation. In all cases the ratio X_0/X had to be raised to the fourth power, so small deviations from the straight line were to be expected. These were, however, well within the limit of experimental error.

Satellite Paradox

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As an artificial satellite gradually descends in a near-circular orbit, its linear velocity increases. Its acceleration in the direction of its motion is found to be the same as if the air drag force, reversed, were pushing the satellite.

THE term "satellite paradox" is somtimes applied to the fact that the presence of air drag results in an increase of velocity of a satellite traveling in a near-circular orbit. It is the purpose here to explain the satellite paradox by deriving the relationships among velocity, altitude, and air drag, for a gradually descending satellite in a near-circular orbit. Supplementary to this purpose, the relationships among potential energy, kinetic energy, and dissipated energy, are developed.

The principal assumption made in the present analysis is that the rate of change of the descent angle of the satellite is very small, as is well-known from actual observations. The approach used herein stems from that of King-Hele.[1]

The notation used is as follows: g, gravitational constant at the surface of the earth; R, radius of the earth (assumed spherical); r, radius from the center of the earth to the satellite; m, mass of the satellite; w, gravitational force on the satellite; D, drag force (air resistance); θ, descent angle (radians), measured from the local horizontal; ϕ, angle between r and a fixed reference line; v, satellite velocity at radius r; and v_c, satellite velocity in a stable circular orbit of radius r.

[1] D. G. King-Hele, J. Brit. Interplanet. Soc. 15, 314 (1956).

CONDITIONS IN A CIRCULAR ORBIT

First, let us consider a satellite traveling in a zero-drag circular orbit. The following conditions then prevail:

$$\text{Centrifugal force} = mv_c^2/r. \quad (1)$$

$$\text{Gravitational force} = w = mgR^2/r^2. \quad (2)$$

Equating centrifugal force to gravitational force gives

$$mv_c^2/r = mgR^2/r^2,$$

from which

$$v_c^2 = gR^2/r. \quad (3)$$

From Eq. (3) it will be found convenient in the following analysis, at times, to substitute v_c^2 for gR^2/r, or vice versa.

CONDITIONS IN A DESCENDING PATH

Now, consider a satellite traveling in a near-circular orbit, in the thin air prevailing at high altitudes. This orbit, then, will be a spiral with an instantaneous descent angle θ, measured from the local horizontal. Figure 1 shows the geometry involving the velocity, the gravitational force, and the drag force. The drag force is assumed exactly opposite to the direction of satellite velocity, which is a close approximation.

A summation of external forces on the satellite,

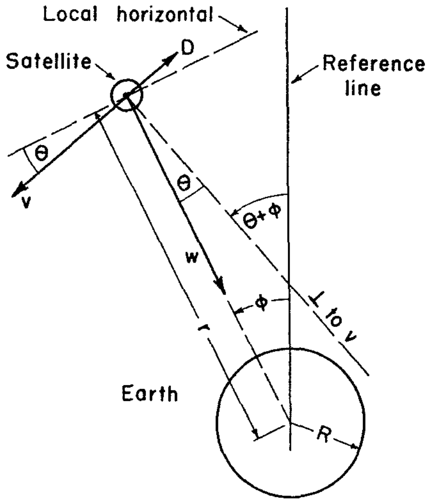

FIG. 1. Satellite dynamics.

taken in the direction of velocity, gives

$$w \sin\theta - D = ma_1 = m(dv/dt), \tag{4}$$

where a_1 is the acceleration component in the direction of v.

A summation of external forces on the satellite, taken perpendicular to the velocity, gives

$$w \cos\theta = ma_2, \tag{5}$$

where a_2 is the acceleration component perpendicular to v.

$$a_2 = v^2/r = v(v/r_c) = v\omega_c = v\frac{d}{dt}(\phi+\theta)$$

$$= v(d\phi/dt + d\theta/dt), \tag{6}$$

where r_c = radius of path curvature (perpendicular to v), and ω_c = angular velocity of r_c, $= d/dt(\phi+\theta)$. The component of velocity v along radius r is

$$dr/dt = -v \sin\theta. \tag{7}$$

The component of v perpendicular to radius r is

$$r(d\phi/dt) = v \cos\theta. \tag{8}$$

From Eq. (8) there follows

$$1/dt = (v \cos\theta)/(rd\phi), \tag{9}$$

so that

$$d\phi/dt = d\phi(v \cos\theta)/(rd\phi) = (v \cos\theta)/r, \tag{10}$$

and

$$d\theta/dt = d\theta(v \cos\theta)/(rd\phi) = (d\theta/d\phi)(v \cos\theta)/r. \tag{11}$$

Substituting Eqs. (10) and (11) in Eq. (6) gives

$$a_2 = v[(v \cos\theta)/r + (d\theta/d\phi)(v \cos\theta)/r]$$
$$= (v^2 \cos\theta)(1 + d\theta/d\phi)/r. \tag{12}$$

Substituting Eq. (12) in Eq. (5) gives

$$w \cos\theta = ma_2 = m(v^2 \cos\theta)(1 + d\theta/d\phi)/r, \tag{13}$$

from which

$$(mv^2/r)(1 + d\theta/d\phi) = w = mgR^2/r^2, \tag{14}$$

and

$$v^2 = (gR^2/r)/(1 + d\theta/d\phi) = v_c^2/(1 + d\theta/d\phi). \tag{15}$$

In Eq. (15), since it is known that the rate of change of the descent angle θ is very small, only very slight error will result from assuming that

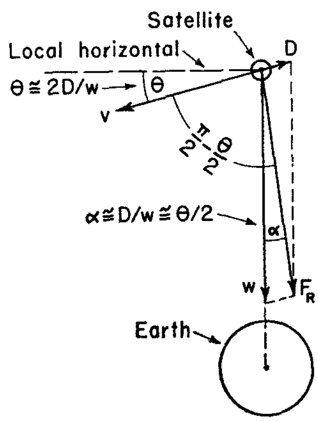

FIG. 2. Sketch showing angle between F_R and V.

$d\theta/d\phi$ is negligible. (Even if the descent angle changed by 3 deg during each trip around the earth, $d\theta/d\phi$ would be only about 3/360, or less than 0.01.) Then, assuming that $d\theta/d\phi$ is negligible compared to unity, Eq. (15) becomes

$$v^2 = v_c^2, \text{ very nearly.} \tag{16}$$

Thus, when the satellite has descended to any radius r, its velocity is practically identical to the velocity of a satellite in a stable circular orbit of that radius. The above derivation of this fact has not assumed any limitation to the magnitude of the descent angle, although it is usually very small. The next step will be to determine the descent angle θ as a function of the drag force D. From Eq. (7) comes

$$1/dt = (-v \sin\theta)/dr. \tag{17}$$

Substituting Eq. (17) in Eq. (4) gives

$$w \sin\theta - D = m(dv/dt) = m(dv)(-v \sin\theta)/dr$$
$$= (vdv/dr)(-m \sin\theta). \quad (18)$$

From Eq. (16) and Eq. (3) comes

$$v = v_c = (gR^2/r)^{\frac{1}{2}}, \quad (19)$$

from which

$$vdv/dr = v_c dv_c/dr = v_c \frac{d}{dr}(gR^2/r)^{\frac{1}{2}}$$
$$= v_c(gR^2)^{\frac{1}{2}}(-1/2)(r^{-\frac{3}{2}})$$
$$vdv/dr = [-v_c(gR^2/r)^{\frac{1}{2}}]/2r = -v_c^2/2r. \quad (20)$$

Substituting Eq. (20) in Eq. (18) gives

$$w \sin\theta - D = (-v_c^2/2r)(-m \sin\theta)$$
$$= (-gR^2/2r^2)(-m \sin\theta)$$
$$= (mgR^2/r^2)(\sin\theta)/2$$
$$w \sin\theta - D = (w/2)(\sin\theta), \quad (21)$$

from which

$$(\sin\theta)(w - w/2) = D = (\sin\theta)(w/2),$$

and

$$\sin\theta = 2D/w. \quad (22)$$

Thus, the descent angle θ (whether small or not), is such that its sine is equal to twice the ratio of the drag force to the gravitational force, at any radius r. This fact is now to be used in determining the rate of increase of satellite velocity as a function of the drag force.

Substituting Eq. (22) in Eq. (4) gives

$$w \sin\theta - D = w(2D/w) - D = m(dv/dt), \quad (23)$$

from which

$$m(dv/dt) = D. \quad (24)$$

Thus, the rate of increase of velocity of the satellite is found to be directly proportional to the drag force; and in fact this acceleration is the same as if the drag force, reversed, were the sole accelerating force.

It seems paradoxical, indeed, that the drag force should appear to accelerate rather than decelerate the satellite. Actually, of course, it is gravity rather than drag that accelerates the satellite. The drag force, however, produces a descent angle θ such that the resultant of the gravitational force and the drag force has a component in the direction of the velocity.

Figure 2 shows the angular relationship between the resultant force on the satellite and the velocity vector. To simplify evaluation of the angle between the resultant force and the velocity, Fig. 2 has been shown with the assumption that the descent angle θ is small enough so that $\sin\theta$ can be taken equal to θ. It is seen that the angle between the resultant force F_R and the velocity vector is less than a right angle by the amount $\theta/2$ or D/w, in radian measure. Thus, the greater the air drag force, the greater the component of resultant force along the direction of the satellite velocity.

ENERGY RELATIONSHIPS

It can be shown that, as the satellite descends, half of its loss in potential energy appears as an increase in kinetic energy, the other half dissipated as heat by the drag force. Toward this purpose, consider the satellite as it descends along its spiral path from radius r_1 to radius r_2. From the previous analysis, it is proper to consider that the velocity v at any radius is equal to the velocity of a satellite in a stable circular orbit of that radius. Then:

Gain in kinetic energy

$$= mv_2^2/2 - mv_1^2/2 = (m/2)(v_2^2 - v_1^2). \quad (25)$$

Loss in potential energy

$$= \int_{r_1}^{r_2} wdr = \int_{r_1}^{r_2} (mgR^2/r^2)dr$$
$$= mgR^2[-1/r]_{r_1}^{r_2} = -mgR^2(1/r_2 - 1/r_1)$$
$$= -m(gR^2/r_2 - gR^2/r_1) = -m(v_2^2 - v_1^2). \quad (26)$$

Comparing Eqs. (25) and (26), it is seen that the gain in kinetic energy is half the loss in potential energy. The other half of the lost potential energy was dissipated by the drag force.