A Paradox Concerning the Energy of A Dipole in a Uniform External Field

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1 Problem

A well-known result of electrostatics is that the energy of an electric dipole of moment $\mathbf{m}$ in and external electric $\mathbf{E}_0$ is given by

$$U_{\rm int} = -\mathbf{m} \cdot \mathbf{E}_0. \tag{1}$$

This expression can, for example, be deduced from the general prescription for the energy of interaction of a charge density ρ in and external electric potential ϕ_0, namely that

$$U_{\rm int} = \int \rho \phi_0 \, d{\rm Vol}. \tag{2}$$

A useful abstraction from a charge distribution with a net dipole moment is the concept of a point dipole consisting of a pair of charges $\pm q$ separated by a small distance $\mathbf{d}$ such that we may take the limit as $d \to 0$ and $q \to \infty$ while the product $qd = m$ remains constant. Then, the electric field of the point dipole can be written (in Gaussian units) as

$$\mathbf{E} = \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} - \frac{4\pi}{3} \mathbf{m} \, \delta^3(\mathbf{r}), \tag{3}$$

where the position vector $\mathbf{r}$ is measured with respect to the center of the point dipole. The meaning of the delta function in eq. (3) is discussed, for example, in sec. 4.1 of [1].

An important contribution of Faraday and Maxwell to electrodynamics was their emphasis on the electromagnetic fields as primary physical concepts. A consequence is that the electrical energy of a system can be calculated as an integral of the electric field rather than of charges and potentials as in eq. (2). In particular, the energy of interaction of charges that create a field $\mathbf{E}$ with an external electric field $\mathbf{E_0}$ can be written

$$U_{\rm int} = \int \frac{\mathbf{E} \cdot \mathbf{E}_0}{4\pi} d{\rm Vol}. \tag{4}$$

The paradox concerns the use of eq. (4) for the interaction energy of a point dipole (3) in a *uniform* external electric field. In this case, we find

$$U_{\rm int} = \frac{\mathbf{E}_0}{4\pi} \cdot \int \mathbf{E} \, d{\rm Vol} = -\frac{1}{3} \mathbf{m} \cdot \mathbf{E}_0, \tag{5}$$

in that the angular integral of $3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}$ at a fixed radius r vanishes.

2 Solution

The paradox may have something to do with the use of an idealized point dipole. So we first consider the case of a dipole consisting of a pair of charges $\pm q$ separated by a finite distance $\mathbf{d}$, where $\mathbf{m} = q\mathbf{d}$ is the dipole moment. The electric field of the dipole can be written as

$$\mathbf{E} = \mathbf{E}_q + \mathbf{E}_{-q}. \tag{6}$$

Using this in eq. (4), we calculate the interaction energy of this dipole in a uniform external field $\mathbf{E}_0$ to be

$$U_{\rm int} = \frac{\mathbf{E}_0}{4\pi} \cdot \int \mathbf{E}\ d{\rm Vol} = \frac{\mathbf{E}_0}{4\pi} \cdot \left(\int \mathbf{E}_q\ d{\rm Vol} + \int \mathbf{E}_{-q}\ d{\rm Vol} \right) = 0, \tag{7}$$

since the volume integral of field of charge q is surely equal and opposite to that of charge $-q$.

We now appear to be in even greater difficulty than before, with Maxwell's expression for field energy predicting that there is no interaction energy for a real dipole in a uniform field!

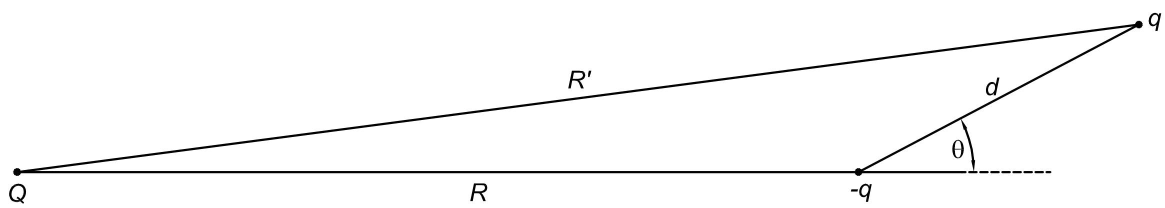
However, on reflection, we realize that the vanishing of the interaction energy in eq. (7) depends on the cancelation of large quantities that are far separated in space, and in reality the cancelation may not be exact. For example, the volume integral of E_z due to charge q can be calculated in a spherical coordinate system (r, θ, φ) centered on the charge as

$$\int E_z\ d{\rm Vol} = 2\pi \int_0^\infty \int_{-1}^1 r^2\ dr\ d\cos\theta\ \frac{q\cos\theta}{r^2} = 2\pi q \int_0^\infty \int_{-1}^1 \cos\theta\ dr\ d\cos\theta. \tag{8}$$

The integrand has the same value in any element $dr\ d\cos\theta$ at a fixed angle θ independent of r. The contributions to the integral from elements on the other side of the universe is just as large as those near to the charge. While the integral (7) is zero mathematically, we cannot say this is true in the physical universe with confidence. In particular, we see that the interaction energy vanishes only if the external field is uniform throughout the entire universe. It is not sufficient to suppose that the external field in uniform across the (small) dipole.

Indeed, the external field $\mathbf{E}_0$ is created by some set of charges suitably remote from the dipole. It suffices to imagine this field as being created by a single charge Q at large distance R from the dipole such that $Q = E_0 R^2$. While charge Q creates a field $\mathbf{E}_Q$ that is essentially uniform near the dipole, this field is by no means uniform throughout the entire universe, and eq. (7) will no longer yield zero for the interaction energy.

We verify this using the geometry shown in the figure below, where $R' = \sqrt{R^2 + d^2 + 2Rd\cos\theta} \approx R[1 + (d/R)\cos\theta]$.



In the Appendix we confirm that Maxwell's expression for field energy can be used to calculate the interaction energy $U_{12} = qq'/D$ between a pair of point charges separated by distance D. Therefore, we calculate the interaction energy of the "external" field due to charge Q with the dipole shown in the figure to be

$$\begin{aligned} U_{\rm int} &= \int \frac{\mathbf{E}_0 \cdot \mathbf{E}}{4\pi}\, d{\rm Vol} = \int \frac{\mathbf{E}_0 \cdot \mathbf{E}_q}{4\pi}\, d{\rm Vol} + \int \frac{\mathbf{E}_0 \cdot \mathbf{E}_{-q}}{4\pi}\, d{\rm Vol} = \frac{qQ}{R'} - \frac{qQ}{R} \\ &\approx \frac{qQ}{R}\left(\frac{1}{1 + (d/R)\cos\theta} - 1\right) = -\frac{qdQ\cos\theta}{R^2} = -mE_0\cos\theta = -\mathbf{m}\cdot\mathbf{E}_0. \end{aligned} \tag{9}$$

Thus, careful attention to the contributions at large distances to Maxwell's expression (4) for the interaction energy restores agreement with the elementary result for a dipole in a "uniform" external field.

However, we have not completely resolved the paradox. We now see that the interaction energy of a "uniform" external field $\mathbf{E}_0$ with the first term in eq. (3) for the field of a point dipole $\mathbf{m}$ will give $-\mathbf{m}\cdot\mathbf{E}_0$ rather than zero. But the use of the second term in eq. (3) spoils our success.

Further, we recall that in the case of a magnetic dipole $\mathbf{m}$, *i.e.*, a tiny loop of current, the magnetic field can be written (see, for example, sec. 5.6 of [1])

$$\mathbf{B} = \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} + \frac{8\pi}{3}\mathbf{m}\,\delta^3(\mathbf{r}), \tag{10}$$

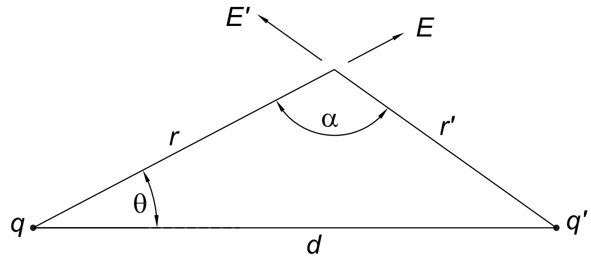
while the interaction energy of a magnetic dipole in a "uniform" external magnetic field $\mathbf{B}_0$ is $U_{\rm int} = -\mathbf{m}\cdot\mathbf{B}_0$. Maxwell's calculation of the interaction energy of a magnetic dipole with the external field yields the desired result using only the first term of eq. (10). Use of the second term again spoils the agreement, but in a different way than for the case of an electric dipole.

The use of the delta-function terms in eqs. (3) and (10) appears not to be justified in a classical context, particularly if one is concerned with calculation of energy. However, in quantum mechanics these delta functions are needed in the proper Hamiltonian for the interaction of dipoles with other fields. See, for example, sec. 5.7 of [1].

3 Appendix: The Interaction Energy of Two Point Charges

We verify that eq. (4) can be used to calculate the interaction energy $U_{12} = qq'/d$ of two point charges q and q' that are separated by distance d, taking $\mathbf{E}$ to be the field of charge q and $\mathbf{E}_0 = \mathbf{E}'$ to be that of charge q'.

We calculate in a spherical coordinate system (r, θ, φ) whose origin is at charge q and whose z axis points towards charge q', as shown in the figure below.



By the law of cosines we have

$$r' = \left(r^2 + d^2 - 2rd\cos\theta\right)^{1/2}, \tag{11}$$

and also

$$\cos\alpha = \frac{r^2 + r'^2 - d^2}{2rr'} = \frac{r - d\cos\theta}{r'}. \tag{12}$$

The interaction energy of the two charges is

$$\begin{aligned} U_{12} &= \int \frac{\mathbf{E}\cdot\mathbf{E}'}{4\pi} d\mathrm{Vol} = \int \frac{qq'\cos\alpha}{4\pi r^2 r'^2} d\mathrm{Vol} = \frac{qq'}{2}\int_0^\infty dr \int_{-1}^{1} d\cos\theta \frac{r - d\cos\theta}{(r^2 + d^2 - 2rd\cos\theta)^{3/2}} \\ &= \frac{qq'}{2}\int_0^\infty dr \left[\left(\frac{1}{|r-d|} - \frac{1}{r+d}\right)\left(\frac{1}{2d} - \frac{d}{2r^2}\right) - \frac{1}{2r^2 d}\left(|r-d| - r - d\right)\right] \\ &= \frac{qq'}{2}\int_0^d dr \left[\left(\frac{1}{d-r} - \frac{1}{r+d}\right)\frac{r^2 - d^2}{2r^2 d} + \frac{1}{rd}\right] + \frac{qq'}{2}\int_d^\infty dr \left[\left(\frac{1}{r-d} - \frac{1}{r+d}\right)\frac{r^2 - d^2}{2r^2 d} + \frac{1}{r^2}\right] \\ &= qq'\int_d^\infty \frac{dr}{r^2} = \frac{qq'}{d}. \end{aligned} \tag{13}$$

Note how the contribution to the energy U_{12} at distances $r < d$ vanishes, and the energy is accounted for in Maxwell's view entirely by the contribution for $r > d$, *i.e.*, at relatively large distances.

4 References

[1] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999).