

# Diffraction as a Consequence of Faraday's Law

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## 1 Problem

A linearly polarized plane electromagnetic wave of frequency  $\omega$  is normally incident on an opaque screen with a square aperture of edge  $a$ .

Show that the wave has a longitudinal magnetic field once it has passed through the aperture by an application of Faraday's Law to a loop parallel to the screen, on the side away from the source. Deduce the ratio of longitudinal to transverse magnetic field, which is a measure of the diffraction angle.

## 2 Solution

Consider a linearly polarized wave with electric field  $\mathbf{E}_x e^{i(kz - \omega t)}$  incident on a perfectly absorbing screen in the plane  $z = 0$  with a square aperture of edge  $a$  centered on the origin. We apply the integral form of Faraday's Law to a semicircular loop with its straight edge bisecting the aperture and parallel to the transverse electric field  $\mathbf{E}_x$ , as shown in the figure. The electric field is essentially zero close to the screen on the side away from the source. Then, at time  $t = 0$ ,

$$\oint \mathbf{E} \cdot d\mathbf{l} \approx E_x a \neq 0. \quad (1)$$

If the loop were on the source side of the screen, the integral would vanish.

Faraday's Law tells us immediately that the time derivative of the magnetic flux through the loop is nonzero. Hence, there must be a nonzero longitudinal component,  $B_z$ , to the magnetic field, once the wave has passed through the aperture.

In Gaussian units,

$$B_y a = E_x a \approx \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c} \frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} \approx -\frac{1}{c} \frac{dB_z}{dt} \frac{a^2}{2}, \quad (2)$$

where  $B_z$  is a characteristic value of the longitudinal component of the magnetic field over that half of the aperture enclosed by the loop. The longitudinal magnetic field certainly has time dependence of the form  $e^{-i\omega t}$ , so  $dB_z/dt = -i\omega B_z = -2\pi ic B_z/\lambda$ , and eq. (2) leads to

$$\frac{B_z}{B_y} \approx -\frac{i\lambda}{\pi a}. \quad (3)$$

By a similar argument for a loop that enclosed the other half of the aperture,  $B_z/B_y \approx i\lambda/\pi a$  in that region;  $B_z = 0$  in the plane  $y = 0$ .

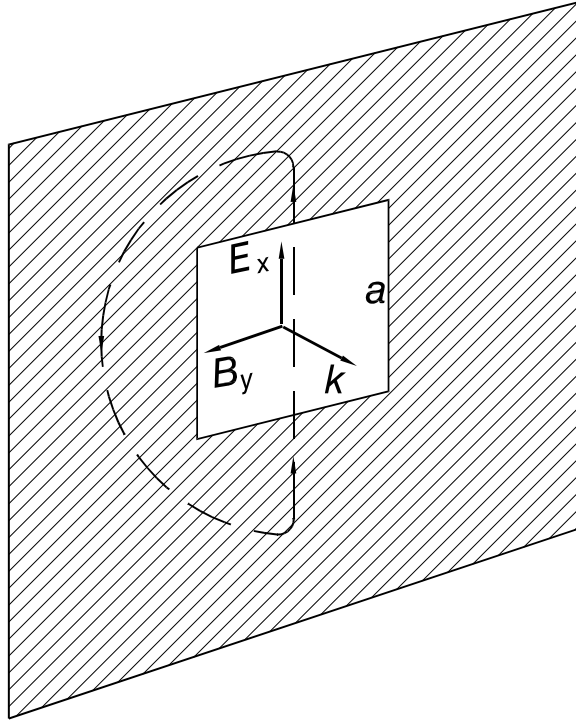


Figure 1: A screen with a square aperture of edge  $a$  is illuminated by a linearly polarized electromagnetic wave. The imaginary loop shown by the dashed curve lies close to the screen, on the side away from the source, and so is partly in the shadow of the wave.

We see that the wave is no longer a plane wave after passing through the aperture, and we can say that it has been diffracted as a consequence of Faraday's Law.

This argument emphasizes the fields near the aperture. A detailed understanding of the fields far from the aperture requires more than just Faraday's Law. A simplified analysis is that the magnitude of the ratio (3) is a measure of the spread of angles of the magnetic field vector caused by the diffraction, and so in the far zone the wave occupies a cone of characteristic angle  $\lambda/\pi a$ .

### 3 Comments

Using the fourth Maxwell equation including the displacement current, we can make an argument for diffraction of the electric field similar to that given above for the magnetic field.

After the wave has passed through the aperture of size  $a$ , it is very much like a wave that has been brought to a focus of size  $a$ . Hence, we learn that near the focus  $(x, y, z) = (0, 0, 0)$  of a linearly polarized electromagnetic wave with  $\mathbf{E} = E\hat{x}$  and propagating in the  $z$  direction, there are both longitudinal electric and magnetic fields, and that  $E_z$  and  $B_z$  are antisymmetric about the planes  $x = 0$  and  $y = 0$ , respectively.

Also, eq. (3) indicates that near the focus the longitudinal and transverse fields are  $90^\circ$

out of phase. Yet, far from the focus, the transverse and longitudinal fields become in phase, resulting in spherical wavefronts that extend over a cone of characteristic angle  $\lambda/\pi a$ . For this to hold, the longitudinal and the transverse fields must experience phase shifts that differ by  $90^\circ$  between the focal point and the far zone.

It is only a slight leap from the present argument to conclude that the transverse fields undergo the extra phase shift. This was first deduced (or noticed) by Gouy [1] in 1890 via the Huygens-Kirchhoff integral [2]. The latter tells us that the secondary wavelet  $\psi$  at a large distance  $r$  from a small region of area  $A$  where the wave amplitude is  $\psi_0 e^{-i\omega t}$  is

$$\psi = \frac{k\psi_0 A}{2\pi i} \frac{e^{i(kr-\omega t)}}{r} = \frac{k\psi_0 A}{2\pi} \frac{e^{i(kr-\omega t-\pi/2)}}{r}. \quad (4)$$

The possibly mysterious factor of  $i$  in the denominator of the Huygens-Kirchhoff integral implies a  $90^\circ$  phase shift between a focus and the far field of a beam of light. Here, we have seen that this phase shift can also be considered as a consequence of Faraday's Law.

## 4 References

- [1] A.E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986), sec. 17.4.
- [2] L. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon Press, Oxford, 1975), sec. 59.