

Energy, Momentum and Stress in a Belt Drive

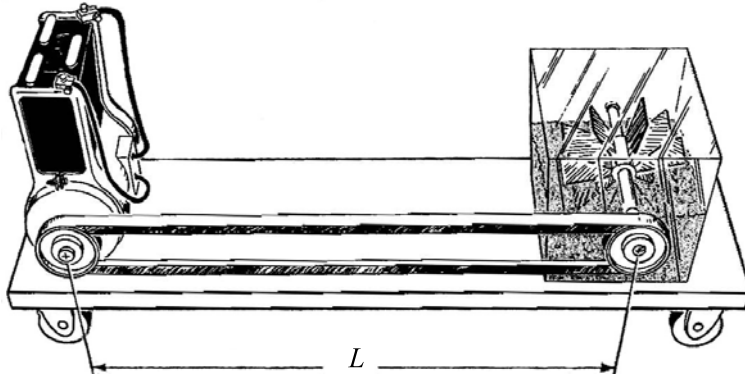
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(October 20, 2007)

1 Problem

Problem 65, p. 147, of the first edition of *Spacetime Physics* by E.F. Taylor and J.A. Wheeler [1] showed the figure below with the caption “Transfer of mass without net transfer of particles or radiation”.



A battery drives a motor that in turn drives a paddlewheel via a belt drive. The paddlewheel turns in a water bath that heats up as a consequence. Hence, energy E is transferred from the battery to the water bath, and as a result the mass of the battery decreases by E/c^2 while the mass of the water increases by the same amount, where c is the speed of light.

How does this energy get from the battery to the water? Do observers in different frames of reference agree as to the path of the energy flow?

Discuss the momentum in this system, supposing that the apparatus rolls without friction on a horizontal table.

The distance between the pulleys is L . The velocity of the belt relative to the baseplate of the apparatus is v and the tension in the tight side of the belt is T . The pulleys and the belt have teeth such that the belt cannot slip, and the tension of the belt is zero on the loose side. The motor is mounted on vibration dampers such that no oscillations of the frequency of the motor or higher are transmitted to the baseplate.

To avoid possible ambiguities as to the role of the loose side of the belt, consider also a configuration in which a string of mass m and length $l \gg L$ is initially wrapped around the pulley attached to the paddlewheel. The string is also tied to the motor pulley, and the motor is operated for a time $t = l/v$, after which time the string is wrapped around the motor pulley.

To support the tension T , both the string and belt must be elastic media, for which the spring constant of the stretched portion is k .

2 Solution

This problem is a variant of Einstein’s argument in which he first deduced the equivalence of energy and mass, $E = mc^2$, by consideration of a system in which energy was transmitted from one side of a close system to another by light waves [2]. Such examples (which include the present case), may also contain components of the mechanical momentum that are of order $1/c^2$, which are sometimes labeled as “hidden” momentum [3, 4, 5, 6, 8, 7, 9, 10, 11].

The present example involves a flow of energy which is not associated with a net motion of massive particles, nor with a flow of readily identifiable electromagnetic radiation. Yet, something must be flowing from the motor to the paddlewheel that carries energy along with it.

In the rest frame of the baseplate of the system it is plausible that the energy flow is within the taut side of the belt, which, however, moves towards the motor rather than towards the paddlewheel. So, we must consider how energy flow can be opposite to the direction of motion of a medium. Furthermore, in the rest frame of the taut side of the belt, we must consider whether the energy still appears to flow down the belt, or whether it flows elsewhere in the system.

Thus, the problem also illustrates the issue of the relativity of energy flow.

The technical complexity of elastic belt drives, in which necessary slippage at the drive pulleys [12, 13] tends to excite stress waves [14, 15], perhaps obscures the relativity of energy flow in the present example. Therefore, the author has prepared two other examples [16, 17], in which the relativity of energy flow is illustrated more crisply.

Here we will ignore possible sound waves in the belt/string,

2.1 The Pulleys Are Connected by a String

We first consider the slightly simpler case that the pulleys are connected by a string of length l whose motion is only from the paddlewheel pulley to the motor pulley.

2.1.1 “Hidden” Momentum Arising from the Mass Equivalence of Energy

The tension in the string is T and the velocity of the string relative to the baseplate of the system is v . Hence, the rate of energy being transferred from the motor to the paddlewheel is

$$\frac{dE}{dt} = vT, \quad (1)$$

according to an observer at rest with respect to the baseplate.

The motor is operated for a total time $t = l/v$, so the total energy transferred from the motor to the paddlewheel and water bath is

$$E = \frac{dE}{dt}t = lT, \quad (2)$$

which is independent of the velocity v of the string.

Following Einstein, we note that associated with this energy transfer is a transfer of mass,

$$\Delta m = \frac{E}{c^2} = \frac{lT}{c^2}, \quad (3)$$

from the battery to the water pool.

If the mass of the entire system on the rolling baseplate is M , and the system was initially at rest with respect to the lab frame, then the center of mass of the system remains at rest in that frame. To compensate for the small transfer of mass from the battery to the water pool, the rest of the system moves a distance

$$x_1 = \frac{\Delta m}{M}L = \frac{lT}{Mc^2}L \quad (4)$$

towards the motor during the time $t = l/v$ that the motor was operated. To accomplish this motion the baseplate has a velocity

$$v_{B1} = \frac{x_1}{t} = \frac{LT}{Mc^2}v \quad (5)$$

relative to the lab frame during this time, and the associated momentum of the system (with respect to the lab frame) is

$$P_1 = Mv_{B1} = \frac{LTv}{c^2}. \quad (6)$$

Since the total momentum of the system with respect to the lab frame is zero, there must be an additional momentum

$$P'_1 = -\frac{LTv}{c^2} \quad (7)$$

somewhere in the system, where the minus sign indicates that the direction of propagation is from the motor to the paddlewheel. Because it will be difficult to locate this momentum, we might describe it as “hidden”.

2.1.2 Momentum of the Moving String

An obvious place to look for the “hidden” momentum is in the string that moves from the paddlewheel pulley to the motor pulley. However, the sign of the momentum associated with the moving string is the same as that of eq. (6). Indeed, during time t the mass m of the string moves distance L relative to the rolling baseplate, and by the same argument that led to eq. (6), the system must have moved a distance

$$x_2 = -\frac{m}{M}L \quad (8)$$

towards the water pool during the time $t = l/v$ that the motor was operated. To accomplish this motion the baseplate had a velocity

$$v_{B2} = \frac{x_2}{t} = -\frac{mL}{Ml}v, \quad (9)$$

and the associated momentum of the system was

$$P_2 = Mv_{B2} = -\frac{mLv}{l}, \quad (10)$$

both with respect to the lab frame. The momentum of the moving string was

$$P'_2 = -P_2 = \frac{mLv}{l} \gg P_1. \quad (11)$$

We have not yet identified the location of the tiny relativistic momentum (7), which can be called a “hidden” momentum.

2.1.3 Can We Identify the Path of the Flow of Energy?

The “hidden” momentum (7) is related to the energy transfer from the motor to the water pool, which brings us to the question of the path of that energy transfer. We ignore the possibilities of electromagnetic radiation through the air around the apparatus, of convective heat transfer in the air, and of thermal conduction in the string or baseplate.¹ The remaining options are that the energy travels from the motor to the water pool via the string, or via the baseplate of the system, or both.

If the moving energy could be localized, then according to Einstein there will be a localized increase in mass. Classical observers could in principle detect such mass increases, however tiny, and so all classical observers should agree as to the path of the energy flow.

The stretched string stores elastic energy, and to observer for whom the string is in motion, there is a corresponding flow of flow of energy. However, the tension in the string also results in a compression of the baseplate, so the latter also stores elastic energy. Then, according to observers for whom the baseplate is in motion, there is also a flow of energy in the baseplate. The elastic energies stored in the string and in the baseplate are both associated with small relativistic increases in their masses, whether or not an observer associates an energy flow with these increments of mass/energy.

The path of steady flow of energy can and will be different for different observers.

2.1.4 Energy Flow and (Hidden) Momentum Density in a Moving Elastic Medium

In a related example [16] we illustrate the flow of energy and the associated density of momentum in an idealized 1-dimensional gas under pressure. In this case, the microscopic motion of the molecules of a gas that also has a bulk motion affords a satisfactory mechanical model of the energy flow in the direction of the bulk motion.

The case of a moving elastic medium under tension cannot be modeled as a gas or liquid, since these media do not support tension. The forces that bind the atoms in an elastic medium are electromagnetic in nature. However, an electrostatic model of the elastic medium will not suffice, since, as noted by Earnshaw [18], there is no stability for (charged) particles subject only to forces associated with an (electro)static potential.

Whatever the details of a microscopic description of an elastic medium, the total electromagnetic interaction energy among constituent atoms is negative. This interaction energy is at a minimum when the medium is unstressed, and is larger but still negative for either a compressive or tensile stress.

In a naïve model of an elastic medium at rest, the electromagnetic interaction energy is electrostatic. If the medium has a bulk motion with velocity $v \ll c$, then the (negative) electrostatic interaction energy is the same, but there is also a magnetic interaction energy of order v^2/c^2 that is negative of the medium is under tension and positive if it is under compression. This magnetic interaction energy corresponds implies a change in the mass of the medium, according to Einstein, and so the transport of mass/energy in the direction of motion is decreased in the case of tensile stress and increased for compressive stress. For a

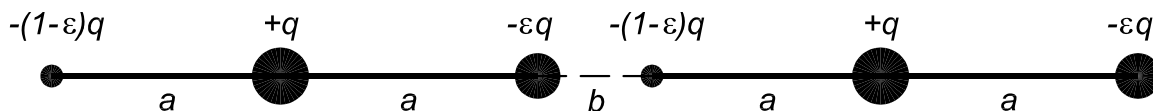
¹Since the water pool becomes warmer than the motor if the latter is 100% efficient, all these effects would transfer energy from the water pool back to the motor.

medium under tension, the decrease in energy flow in the direction of motion is equivalent to a positive energy flow that opposes the motion.

Associated with the apparent flow of magnetic energy for a stressed, moving elastic medium is a density of momentum, which is the “hidden” momentum we have been seeking to identify.

We illustrate the preceding discussion with a simplified model of an elastic medium as consisting of atoms that are electric dipoles aligned along the $+z$ axis. Then, there is an attractive force in the z direction between any pair of dipoles, and the medium is under tension.

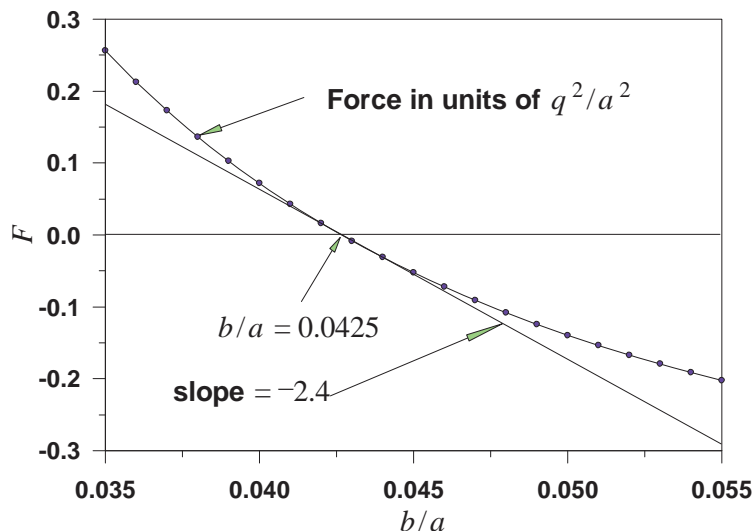
To keep the dipoles from collapsing on one another, we suppose that each atom has a small electric quadrupole moment as well. In more detail, we suppose the atomic nucleus has charge $+q$, and the atomic electrons are represented by a charge $-\epsilon q$ located at distance $+a$ from the nucleus and by a second charge $-(1-\epsilon)q$ located at $-a$ with respect to the nucleus. Neighboring atoms are separated by distance b , as shown in the figure below. The atomic size a is somehow fixed, but the separation b between atoms varies with the stress in the medium.



The basic features of this model are illustrated by a “solid” that consists of only two atoms. The force between these two atoms is

$$F = q^2 \left[\frac{\epsilon(1-\epsilon)}{b^2} - \frac{1}{(a+b)^2} + \frac{2(1-\epsilon+\epsilon^2)}{(2a+b)^2} - \frac{1}{(3a+b)^2} + \frac{\epsilon(1-\epsilon)}{(4a+b)^2} \right]. \quad (12)$$

For $\epsilon = 0.999$ the quadrupole strength is $1/1000$ of the dipole strength, but this is sufficient to provide a nominal equilibrium point (for motion along the dipole axis) at separation $b = 0.0425a$. A plot of the force as a function of b/a is shown below. For separation $b > 0.0425a$ the force between neighboring atoms is attractive and the medium is under tension.



3 Acknowledgment

The author thanks Edwin Taylor and Vladimir Hnizdo for e-conversations about this problem.

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