

Classical Aspects of the Aharonov-Bohm Effect

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1 Problem

In 1926 Fock noted [1, 2, 3] that Schrödinger's equation for an electric charge e of mass m in electromagnetic fields described by potentials $A_\mu = (\phi, \mathbf{A})$ can be written, in Gaussian units with c as the speed of light,

$$\frac{(-i\hbar\mathbf{D})^2}{2m}\psi = i\hbar D_0\psi, \quad \text{using the "altered" (covariant) derivative } D_\mu = \partial_\mu - ieA_\mu/\hbar c, \quad (1)$$

which is gauge invariant only if the gauge transformation of the potentials, $A_\mu(x_\nu) \rightarrow A_\mu + \partial_\mu\Omega(x_\nu)$, is accompanied by a phase change of the wavefunction, $\psi(x_\nu) \rightarrow e^{-ie\Omega(x_\nu)/\hbar c}\psi$. Yang and Mills (1954) [4, 5] may have been the first to point out that Fock's argument can be inverted such that a requirement of local phase invariance of the form $\psi(x_\nu) \rightarrow e^{-ie\Omega(x_\nu)/\hbar c}\psi$ implies the existence of an interaction described by a potential A_μ (and charge e) which satisfies gauge invariance and modifies Schrödinger's equation via the altered derivative D_μ . This led to a greater appreciation of the significance of potentials in the quantum realm.

Separately, consideration of possible interference effects in electron microscopy [6] led Aharonov and Bohm [7, 8] to discuss an electron that moves only outside a long solenoid magnet (where $\mathbf{B} = 0$ to a good approximation), and which accumulates a different phase in its wavefunction depending on which side of the magnet it passes. The resulting interference pattern, which depends on the (gauge-invariant) magnetic flux in the solenoid (that can be related to the vector potential \mathbf{A} in whatever gauge is used), has been observed in subsequent experiments [9, 10].¹

The quantum interference effect in the Aharonov-Bohm experiment is impressive, but there are already disconcerting issues in purely classical considerations thereof. It is often remarked that there is no classical effect on an electron that passes outside a long solenoid magnet, where $\mathbf{B}_{\text{solenoid}} = 0$. However, the current density that generates the solenoid field is affected by the magnetic field of the moving electron (even assuming that the electric charge density associated with the current density is zero).

Problem: Deduce the force on a solenoid of radius a about the z -axis that carries azimuthal surface current density $K_\theta = I$ per unit length, when an electron of velocity $\mathbf{v} = v\hat{\mathbf{y}}$ is at position $(x, y, z) = (b, vt, 0)$, where $v \ll c$ and $|b| \gg a$.²

¹This result is often misinterpreted as evidence that the vector potential \mathbf{A} is "observable" in the quantum realm. A better statement is that there exist quantum-electrodynamic effects on the behavior of an electron which moves only in a region of zero external electric and magnetic field, but where the vector potential (in any choice of gauge) is nonzero. Note that the observed result relates directly to the magnetic field $\mathbf{B}_{\text{solenoid}}$ although this field is zero at the electron; the paradox is more that the observed quantum effect seems to be action-at-a-distance (as bothered E-P-R in another context [11]) between the solenoid and the electron.

²Assume that the magnetic field of the electron is not "shielded" by the solenoid, which shielding would imply additional currents that create additional magnetic field external to the solenoid that lead to a force on the moving electron.

That is, Newton’s third law is not obeyed by this configuration!

Issues like this were noted by Ampère in the 1820’s and led him to doubt the existence of isolated, moving electric charges, which view put particle physics on hold for 60 years. Only after Poynting (1884) [12] developed the notion that electromagnetic fields can support a flux of energy (and hence also contain momentum [13, 14, 15]), did physicists have the confidence to reconsider the concept of elementary charged particles.³

In retrospect we note that the issue of apparent violation of Newton’s third law could have been resolved earlier, based on Faraday’s insight that what we now call the vector potential \mathbf{A} (called the “electrotonic state” by Faraday [18]) can be associated with “electromagnetic momentum”, as formulated mathematically by Maxwell [19]. In Gaussian units, the electromagnetic momentum associated with a charge distribution ρ that is immersed in a vector potential \mathbf{A} (in the Coulomb gauge, strictly speaking) is given (for quasistatic motion) by^{4,5}

$$\mathbf{P}_{\text{EM}} = \int \frac{\rho \mathbf{A}}{c} d\text{Vol}. \quad (2)$$

Problem: Use eq. (2) to deduce the electromagnetic momentum of the electron + solenoid when the electron is at $(x, y, z) = (b, vt, 0)$, and from this show that $d\mathbf{P}_{\text{EM}}/dt$ is equal and opposite to the force on the solenoid found previously.

This seems to be a satisfactory resolution to the issue of momentum conservation, but a disconcerting result remains. Suppose the electric charge were at rest; then the electromagnetic momentum (2) is nonzero, while the solenoid is at rest also and seems to contain no net momentum. Hence, we have an example of a system at rest which seems to contain nonzero total momentum!

Peculiarities of this sort were dramatized by Shockley in 1967 [23],⁶ and remain an arcane

³An important first step was taken by Thomson in 1881 [16, 17] based on considerations of kinetic energy of a moving charge.

⁴The Faraday-Maxwell form (2) is a classical effect of the solenoid on the electron, but it does not imply that the vector potential is observable in classical electrodynamics. Rather, we note that it is equivalent to the Poynting-Poincaré form, $\mathbf{P}_{\text{EM}} = \int \mathbf{E} \times \mathbf{B} d\text{Vol}/4\pi c$, as shown, for example, in [21]. While the Faraday-Maxwell form for the electromagnetic momentum suggests that this resides with the electron, the Poynting-Poincaré form suggests that it resides in the solenoid. This classical ambiguity is a preview of the Aharonov-Bohm effect that an electron can be affected by an electromagnetic field even if the latter is zero at the electron.

⁵The Coulomb-gauge vector potential $\mathbf{A}^{(C)}$ is “rotational”, meaning that $\nabla \cdot \mathbf{A}^{(C)} = 0$. In a general gauge, the vector potential can be written (using Helmholtz’ theorem [20]) as $\mathbf{A} = \mathbf{A}_{\text{irr}} + \mathbf{A}_{\text{rot}}$ where $\nabla \times \mathbf{A}_{\text{irr}} = 0$ and $\nabla \cdot \mathbf{A}_{\text{rot}} = 0$. Then, a gauge transformation $\mathbf{A} \rightarrow \mathbf{A} - \nabla\Omega$, $\phi \rightarrow \phi + \partial\Omega/\partial ct$ implies that $\mathbf{A}_{\text{irr}} + \mathbf{A}_{\text{rot}} \rightarrow (\mathbf{A}_{\text{irr}} - \nabla\Omega) + \mathbf{A}_{\text{rot}}$, such that \mathbf{A}_{rot} is actually a gauge-invariant quantity. Note that $\mathbf{A}_{\text{rot}} = \mathbf{A}^{(C)}$, *i.e.*, the rotational part of the vector potential in any gauge is the Coulomb-gauge vector potential.

It is sometimes said that gauge-invariant quantities are “observable”. However, the Coulomb-gauge vector potential $\mathbf{A}^{(C)} = \mathbf{A}_{\text{rot}}$ in general involves instantaneous effects of distant currents, so claims that the Coulomb-gauge vector potential is “observable” are associated with claims that instantaneous action at a distance is also “observable”. This author takes the view that instantaneous action at a distance is not “observable” (even in the quantum realm) and that the Coulomb-gauge vector potential is not “observable”.

See also [22].

⁶These peculiarities were previously noted by J.J. Thomson in 1904 [17, 24, 25, 26], but were little discussed by others for many years. A partial awareness of classical “paradoxes” in the Aharonov-Bohm

aspect of classical physics, where some systems contain “hidden” momentum [27] (such that systems “at rest” indeed have zero total momentum). One can give a semiplausible classical model of the “hidden” momentum as residing in the electrical current in the present example [29]. Perhaps the main significance of the “hidden” momentum for the Aharonov-Bohm effect is to remind us that even in a “classical” view, the electron is “entangled” with the solenoid, although the magnetic field of the solenoid happens to be zero at the location of the electron. While the field of the solenoid has no “classical” effect on the electron, the electron does have a “classical” effect on the solenoid, so the two objects should not be regarded as independent entities. In this context, it should be pleasing, rather than disturbing, that in the quantum realm the solenoid has an effect on the electron.⁷

Extended Problem: Comment also on energy and angular momentum in this system.

In the author’s view, the Aharonov-Bohm effect (and the related debate about the “observability” of potentials [32]) misses the point that the role of the potentials (which must obey gauge invariance), combined with the notion of local phase invariance, is to determine the form of the interactions of elementary particles. It is the nonobservability of the potentials, because they are subject to gauge transformations, which leads the potentials to be included in the altered derivative D_μ , eq. (1), that makes them so important in the development of the theory of elementary particles and fields.⁸

effect appeared in [28].

⁷This theme was developed by Aharonov for the “dual” example in which a loop of current (magnetic dipole) interacts with a line charge parallel to the axis of the loop [30]. See also [31].

⁸For a similar commentary, which includes demonstration that the Aharonov-Bohm effect vanishes in a suitable classical limit, see [33].

2 Solution

2.1 Momentum

The magnetic field \mathbf{B}_e at position \mathbf{x} of an electron of charge $-e$ and velocity \mathbf{v} at position \mathbf{x}_e is (in Gaussian units, and for $v \ll c$)

$$\mathbf{B}_e(\mathbf{x}, \mathbf{x}_e) = \frac{\mathbf{v}}{c} \times \mathbf{E}_e(\mathbf{x}, \mathbf{x}_e) = -\frac{ev \times \mathbf{R}}{cR^3}, \quad \text{where} \quad \mathbf{R} = \mathbf{x} - \mathbf{x}_e. \quad (3)$$

The force of this magnetic field from an electron at $(x, y, z) = (b, vt, 0)$ on a solenoidal (surface) current density $K_\theta(r = a, \theta, z) = I$ per unit length is

$$\begin{aligned} \mathbf{F} &= \int \frac{\mathbf{K} \times \mathbf{B}_e}{c} d\text{Area} \\ &= -\frac{Iev}{c^2} \int_{-\infty}^{\infty} dz \int_0^{2\pi} a d\theta \frac{(-\sin \theta, \cos \theta, 0) \times \{\hat{\mathbf{y}} \times [(a \cos \theta, a \sin \theta, z) - (b, vt, 0)]\}}{[(a \cos \theta - b)^2 + (a \sin \theta - vt)^2 + z^2]^{3/2}} \\ &= -\frac{aIev}{c^2} \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\theta \frac{(-\sin \theta, \cos \theta, 0) \times (z, 0, b - a \cos \theta)}{(a^2 + b^2 + v^2 t^2 - 2ab \cos \theta - 2avt \sin \theta + z^2)^{3/2}} \\ &= -\frac{aIev}{c^2} \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz \frac{[\cos \theta (b - a \cos \theta), \sin \theta (b - a \cos \theta), -z \cos \theta]}{(a^2 + b^2 + v^2 t^2 - 2ab \cos \theta - 2avt \sin \theta + z^2)^{3/2}} \\ &= -\frac{2aIev}{c^2} \int_0^{2\pi} d\theta \frac{[\cos \theta (b - a \cos \theta), \sin \theta (b - a \cos \theta), 0]}{b^2 + v^2 t^2 - 2ab \cos \theta - 2avt \sin \theta + a^2} \\ &\approx -\frac{2aIev}{c^2(b^2 + v^2 t^2)} \int_0^{2\pi} d\theta [\cos \theta (b - a \cos \theta), \sin \theta (b - a \cos \theta), 0] \\ &\quad \left(1 + \frac{2ab}{b^2 + v^2 t^2} \cos \theta + \frac{2avt}{b^2 + v^2 t^2} \sin \theta \right) \\ &= -\frac{2\pi a^2 Iev}{c^2(b^2 + v^2 t^2)} \left(-1 + \frac{2b^2}{b^2 + v^2 t^2}, \frac{2bvt}{b^2 + v^2 t^2}, 0 \right) \\ &= -\frac{2\pi a^2 Iev}{c^2(b^2 + v^2 t^2)} \left(\frac{b^2 - v^2 t^2}{b^2 + v^2 t^2}, \frac{2bvt}{b^2 + v^2 t^2}, 0 \right). \end{aligned} \quad (4)$$

This force is very small, being of order $1/c^2$, and clarification of its possible effect on the system is more of “academic” than practical interest. Note that $\pi a^2 I/c$ is the magnetic moment per unit length along the solenoid

The uniform magnetic field $\mathbf{B}_{\text{solenoid}} = B \hat{\mathbf{z}}$ inside the solenoid has magnitude $B = 4\pi I/c$, as follows from Ampère’s law. This field can also be deduced from a vector potential \mathbf{A} whose only nonzero component in a cylindrical coordinate system (r, θ, z) is $A_\theta(r)$, where $\mathbf{B} = \nabla \times \mathbf{A}$ implies for a loop of radius r that

$$\oint \mathbf{A} \cdot d\mathbf{l} = 2\pi r A_\theta = \int \nabla \times \mathbf{A} \cdot d\text{Area} = \mathbf{B} \cdot d\text{Area} = \frac{4\pi^2 I}{c} \begin{cases} r^2 & (r < a), \\ a^2 & (r > a). \end{cases} \quad (5)$$

Outside the solenoid, the magnetic field is zero (in the limit of an infinite solenoid), while the vector potential (in the Coulomb gauge) can be taken as⁹

$$A_\theta(r > a) = \frac{2\pi a^2 I}{cr}, \quad \mathbf{A}(r > a) = \frac{2\pi a^2 I}{cr^2}(-y, x, 0). \quad (6)$$

According to eq. (2) of Faraday and Maxwell, the system of electron plus solenoid has electromagnetic momentum¹⁰

$$\mathbf{P}_{\text{EM}}^{(1)} = -\frac{e\mathbf{A}(b, vt, 0)}{c} = -\frac{2\pi a^2 I e}{c^2(b^2 + v^2 t^2)}(-vt, b, 0). \quad (7)$$

The time derivative of this is

$$\begin{aligned} \frac{d\mathbf{P}_{\text{EM}}}{dt} &= -\frac{2\pi a^2 I e v}{c^2(b^2 + v^2 t^2)} \left[-1 + \frac{2v^2 t^2}{b^2 + v^2 t^2}, \frac{-2bvt}{b^2 + v^2 t^2}, 0 \right] \\ &= \frac{2\pi a^2 I e v}{c^2(b^2 + v^2 t^2)} \left[\frac{b^2 - v^2 t^2}{b^2 + v^2 t^2}, \frac{2bvt}{b^2 + v^2 t^2}, 0 \right] = -\mathbf{F} = -\frac{d\mathbf{P}_{\text{mech}}}{dt}, \end{aligned} \quad (8)$$

on comparison with eq. (4). Thus,

$$\mathbf{P}_{\text{EM}} = -\frac{e\mathbf{A}(b, vt, 0)}{c} = -\frac{2\pi a^2 I e}{c^2(b^2 + v^2 t^2)}(-vt, b, 0). \quad (9)$$

The time derivative of this is

$$\frac{d\mathbf{P}_{\text{total}}}{dt} = \frac{d\mathbf{P}_{\text{EM}}}{dt} + \frac{d\mathbf{P}_{\text{mech}}}{dt} = 0, \quad (10)$$

and the total momentum of the system is constant in time. The electrical current in the solenoid carries momentum, but naïvely we expect that the total mechanical momentum of a current loop would be zero; however, this is not the case if the current loop is subject to an external electric field, as in the present example.

The unbalanced force of the moving electron on the solenoid serves to change its “hidden” internal mechanical momentum, while the bulk of the solenoid remains at rest as the electron passes by.^{11,12}

⁹Other choices for the vector potential \mathbf{A} are possible. In particular, use of the Poincaré gauge leads to \mathbf{A} that is nonzero only in a region that depends on the (arbitrary) choice of origin [34], so that while it can be said that the charge interacts “locally” with the vector potential, the location of the “local” interaction is not uniquely determined.

¹⁰The form (7) for the electromagnetic momentum does not appear to be gauge invariant. As discussed in sec. 2.3 of [22], the vector potential in eq. (7) should actually be the so-called **rotational** part, \mathbf{A}_{rot} , of the vector potential, which obeys $\nabla \cdot \mathbf{A}_{\text{rot}} = 0$. That is, the rotational part of any vector potential equals the Coulomb-gauge vector potential, as used here, such that eq. (7), properly interpreted, is gauge invariant.

¹¹All this is rather subtle, and apparently not well known, as a paper based on this example was recently published in Phys. Rev. Lett. claiming that the Lorentz force law must be wrong. For discussion by the author of this dismal issue, see [35].

¹²For a discussion of the character of the “hidden” mechanical momentum in a current loop, see [29, 36].

For completeness, we evaluate the electromagnetic momentum according to the Poynting-Poincaré prescription (ignoring the self-field-momentum of the moving electron),

$$\begin{aligned}\mathbf{P}_{\text{EM}}^{(2)} &= \int \frac{\mathbf{E}_e \times \mathbf{B}_{\text{solenoid}}}{4\pi c} d\text{Vol} \approx \frac{1}{4\pi c} \int_{-\infty}^{\infty} \pi a^2 dz \frac{-e(-b, -vt, z)}{(b^2 + v^2 t^2 + z^2)^{3/2}} \times \frac{4\pi I(0, 0, 1)}{c} \\ &= -\frac{\pi a^2 I e(-vt, b, 0)}{c^2} \int_{-\infty}^{\infty} \frac{dz}{(b^2 + v^2 t^2 + z^2)^{3/2}} = -\frac{2\pi a^2 I e(-vt, b, 0)}{c^2 (b^2 + v^2 t^2)},\end{aligned}\quad (11)$$

as in eq. (7).

It turns out [21, 37] there is a third way that the electromagnetic momentum can be computed (for quasistatic examples) based on the electric scalar potential ϕ and the current density \mathbf{J} ,

$$\begin{aligned}\mathbf{P}_{\text{EM}}^{(3)} &= \int \frac{\phi \mathbf{J}}{c^2} d\text{Vol} = \int \frac{\phi_e \mathbf{K}}{c^2} d\text{Area} \\ &= -\frac{Ie}{c^2} \int_{-\infty}^{\infty} dz \int_0^{2\pi} a d\theta \frac{(-\sin \theta, \cos \theta, 0)}{[(a \cos \theta - b)^2 + (a \sin \theta - vt)^2 + z^2]^{1/2}} \\ &= -\frac{aIe}{c^2} \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\theta \frac{(-\sin \theta, \cos \theta, 0)}{(z^2 + b^2 + v^2 t^2 - 2ab \cos \theta - 2avt \sin \theta + a^2)^{1/2}} \\ &\approx -\frac{aIe}{c^2} \int_{-\infty}^{\infty} \frac{dz}{(z^2 + b^2 + v^2 t^2)^{1/2}} \int_0^{2\pi} d\theta (-\sin \theta, \cos \theta, 0) \left(1 + \frac{ab \cos \theta + avt \sin \theta}{z^2 + b^2 + v^2 t^2}\right) \\ &= -\frac{\pi a^2 I e(-vt, b, 0)}{c^2} \int_{-\infty}^{\infty} \frac{dz}{(z^2 + b^2 + v^2 t^2)^{3/2}} = -\frac{2\pi a^2 I e(-vt, b, 0)}{c^2 (b^2 + v^2 t^2)}.\end{aligned}\quad (12)$$

The fact that electromagnetic momentum can be computed several different ways reminds us that even in “classical” systems the subsystems should be regarded as “entangled” rather than “independent”.

2.2 Energy

The system of electron plus solenoid has an interaction field energy,

$$\begin{aligned}U_{\text{EM,int}} &= \int \frac{\mathbf{E}_e \cdot \mathbf{E}_{\text{solenoid}} + \mathbf{B}_e \cdot \mathbf{B}_{\text{solenoid}}}{4\pi} d\text{Vol} \approx \frac{\pi a^2 I}{c} \int_{-\infty}^{\infty} B_e(0, 0, z) dz \\ &= \frac{\pi a^2 I}{c} \int_{-\infty}^{\infty} \frac{evb}{c(b^2 + v^2 t^2 + z^2)^{3/2}} dz = \frac{2\pi a^2 evbI}{c^2 (b^2 + v^2 t^2)} = e \frac{\mathbf{v}}{c} \cdot \mathbf{A}_{\text{sol}} = \int \frac{\mathbf{J}_e \cdot \mathbf{A}_{\text{sol}}}{c} d\text{Vol},\end{aligned}\quad (13)$$

since $\mathbf{E}_{\text{solenoid}} = 0$ and $\mathbf{B}_{\text{solenoid}} = 4\pi I \hat{\mathbf{z}}/c$ with magnetic flux $\Phi_{\text{sol}} = \pi a^2 B_{\text{solenoid}}$. That is, the electromagnetic interaction field energy (which can have either sign) is greatest in magnitude when the electron is closest to the solenoid.

Where is the compensating energy such that the total energy of the (isolated) system is conserved?

In the Aharonov-Bohm effect it is tacitly assumed that the magnetic field of the solenoid does not change with time, and it is typically implied that this is because that field is due to permanent magnetism. For example, the uniform surface current density $K_\theta = I$ considered

on p. 2 could be due to a cylinder of radius a with uniform magnetization density $\mathbf{M} = I \hat{\mathbf{z}}/c$. In this case, there is an additional interaction energy,

$$U_{M,\text{int}} = - \int \mathbf{M} \cdot \mathbf{B}_e d\text{Vol} \approx - \frac{\pi a^2 I}{c} \int_{-\infty}^{\infty} B_e(0, 0, z) dz = -U_{\text{EM,int}}, \quad (14)$$

such that the total interaction energy is zero.¹³

However, it could be that the solenoid is made of a conductive material and the surface currents are maintained constant by a battery. In this case, the battery does work on the currents (and *vice versa*) such that the sum of the field energy and that of the battery remains constant. One way to see this is to take the time derivative of eq. (13),¹⁴

$$\begin{aligned} \frac{dU_{\text{EM,int}}}{dt} &= \int \frac{\partial \mathbf{B}_e}{\partial t} \cdot \frac{\mathbf{B}_{\text{solenoid}}}{4\pi} d\text{Vol} = - \frac{c}{4\pi} \int \nabla \times \mathbf{E}_e \cdot \mathbf{B}_{\text{solenoid}} d\text{Vol} \\ &= - \frac{c}{4\pi} \int \mathbf{E}_e \cdot \nabla \times \mathbf{B}_{\text{solenoid}} d\text{Vol} = - \int \mathbf{J}_{\text{solenoid}} \cdot \mathbf{E}_e d\text{Vol}, \end{aligned} \quad (15)$$

using the identity that $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$. That is the change in the interaction field energy is opposite to the work done by the field \mathbf{E}_e on the currents. To keep the currents constant, the battery must do work on them opposite to that done by \mathbf{E}_e , which means that the change of energy of the battery equals the work done by \mathbf{E}_e on the currents, which is opposite to the change in the field energy.

We might worry that the “hidden” momentum invoked in sec. 2.1 is associated with a “hidden” energy that should be considered here. However, as discussed in [29], “hidden” mechanical momentum can be thought of as arising because the total energy of the charge carriers of the current in an external electric field remains constant; if the electric potential energy of the charge rises, the kinetic energy (and momentum) decreases, etc.

2.3 Angular Momentum

The force (4) of the electron on the solenoid is associated with a torque,

$$\begin{aligned} \boldsymbol{\tau} &= \int \mathbf{r} \times \frac{\mathbf{K} \times \mathbf{B}_e}{c} d\text{Area} \\ &= - \frac{Iev}{c^2} \int_{-\infty}^{\infty} dz \int_0^{2\pi} a d\theta \frac{\mathbf{r} \times (-\sin \theta, \cos \theta, 0) \times \{\hat{\mathbf{y}} \times [(a \cos \theta, a \sin \theta, z) - (b, vt, 0)]\}}{[(a \cos \theta - b)^2 + (a \sin \theta - vt)^2 + z^2]^{3/2}} \\ &= - \frac{aIev}{c^2} \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\theta \frac{\mathbf{r} \times (-\sin \theta, \cos \theta, 0) \times (z, 0, b - a \cos \theta)}{(a^2 + b^2 + v^2 t^2 - 2ab \cos \theta - 2avt \sin \theta + z^2)^{3/2}} \\ &= - \frac{aIev}{c^2} \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz \frac{(a \cos \theta, a \sin \theta, z) \times [\cos \theta(b - a \cos \theta), \sin \theta(b - a \cos \theta), -z \cos \theta]}{(a^2 + b^2 + v^2 t^2 - 2ab \cos \theta - 2avt \sin \theta + z^2)^{3/2}} \end{aligned}$$

¹³A delicacy is that if the solenoid does not move, no work is done on it by the field \mathbf{B}_e , so eq. (14) does not obviously represent a stored energy. However, in case of a permanent solenoid magnet, the meaning of the interaction energy (13) is also doubtful in that one cannot well represent permanent magnetism by a classical current density \mathbf{J} . That the energies (13) and (14) cancel for a permanent solenoid magnet is a “classical” accommodation of an ultimately quantum phenomenon.

¹⁴This argument was suggested by D.J. Griffiths.

$$= -\frac{aIev}{c^2} \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz \frac{(-bz \sin \theta, bz \cos \theta, 0)}{(a^2 + b^2 + v^2 t^2 - 2ab \cos \theta - 2avt \sin \theta + z^2)^{3/2}} = 0. \quad (16)$$

Since the solenoid exerts no force/torque on the electron, the total torque on the system is zero and we expect its angular momentum to be constant in time.¹⁵

The system of uniformly moving electron plus constant-field solenoid contains both field and mechanical angular momentum. The mechanical angular momentum of the electron is constant in time.

It is delicate to compute the field angular momentum, in that the calculation requires assigning a location to the field momentum. The three prescriptions, (7), (11) and (12) which gave the same value for the total field momentum suggest different locations for it, and hence lead to different values for the field momentum.

The field angular momentum can be computed from eq. (7) as

$$\mathbf{L}_{\text{EM}}^{(1)} = \mathbf{r}_e \times \mathbf{P}_{\text{EM}}^{(1)} = -\frac{\mathbf{r}_e \times e\mathbf{A}}{c} = -(b, vt, 0) \times \frac{2\pi a^2 I e(-vt, b, 0)}{c^2(b^2 + v^2 t^2)} = -\frac{2\pi a^2 I e \hat{\mathbf{z}}}{c^2}, \quad (17)$$

which is constant in time. Using eq. (11) we find

$$\begin{aligned} \mathbf{L}_{\text{EM}}^{(2)} &= \int \mathbf{r} \times \frac{\mathbf{E}_e \times \mathbf{B}_{\text{solenoid}}}{4\pi c} d\text{Vol} \\ &\approx \frac{1}{4\pi c} \int_0^a r dr \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz (r \cos \theta, r \sin \theta, z) \times \\ &\quad \frac{-e(r \cos \theta - b, r \sin \theta - vt, z)}{[(r \cos \theta - b)^2 + (r \sin \theta - vt)^2 + z^2]^{3/2}} \times \frac{4\pi I(0, 0, 1)}{c} \\ &= -\frac{Ie}{c^2} \int_{-\infty}^{\infty} dz \int_0^a r dr \int_0^{2\pi} d\theta \frac{[z(r \sin \theta - b), z(r \cos \theta - b), rb \cos \theta + rvt \sin \theta - r^2]}{(z^2 + b^2 + v^2 t^2 - 2rb \cos \theta - 2rvt \sin \theta + r^2)^{3/2}} \\ &\approx -\frac{Ie}{c^2} \int_{-\infty}^{\infty} dz \int_0^a r dr \int_0^{2\pi} d\theta \frac{(0, 0, rb \cos \theta + rvt \sin \theta)}{(z^2 + b^2 + v^2 t^2)^{3/2}} \left(1 + 3 \frac{rb \cos \theta + rvt \sin \theta}{z^2 + b^2 + v^2 t^2}\right) \\ &= -\frac{Ie}{c^2} \int_{-\infty}^{\infty} \frac{dz}{(z^2 + b^2 + v^2 t^2)^{5/2}} \int_0^a r dr 3\pi r^2 (b^2 + v^2 t^2) \\ &= -\frac{\pi a^4 I e(0, 0, 1)}{c^2(b^2 + v^2 t^2)} = -\frac{\pi a^4 I e \hat{\mathbf{z}}}{c^2(b^2 + v^2 t^2)}, \end{aligned} \quad (18)$$

which varies with time. For completeness, we use eq. (12) to obtain

$$\begin{aligned} \mathbf{L}_{\text{EM}}^{(3)} &= \int \mathbf{r} \times \frac{\phi \mathbf{J}}{c^2} d\text{Vol} = \int \mathbf{r} \times \frac{\phi_e \mathbf{K}}{c^2} d\text{Area} \\ &= -\frac{Ie}{c^2} \int_{-\infty}^{\infty} dz \int_0^{2\pi} a d\theta \frac{(a \cos \theta, a \sin \theta, z) \times (-\sin \theta, \cos \theta, 0)}{[(a \cos \theta - b)^2 + (a \sin \theta - vt)^2 + z^2]^{1/2}} \\ &= -\frac{aIe}{c^2} \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\theta \frac{(-z \cos \theta, -z \sin \theta, a)}{(z^2 + b^2 + v^2 t^2 - 2ab \cos \theta - 2avt \sin \theta + a^2)^{1/2}} \end{aligned}$$

¹⁵In the case where the solenoid is a cylinder of uniform magnetization density \mathbf{M} the net torque of the electron's magnetic field on the magnetization is zero (although there is a bending moment).

$$\begin{aligned}
&\approx -\frac{a^2 I e}{c^2} \int_{-\infty}^{\infty} \frac{dz}{(z^2 + b^2 + v^2 t^2)^{1/2}} \int_0^{2\pi} d\theta (0, 0, 1) \left(1 + \frac{ab \cos \theta + avt \sin \theta}{z^2 + b^2 + v^2 t^2} \right) \\
&= -\frac{2\pi a^2 I e (0, 0, 1)}{c^2} \int_{-\infty}^{\infty} \frac{dz}{(z^2 + b^2 + v^2 t^2)^{1/2}} = \mathbf{L}_{\text{EM}}^{(1)} \int_{-\infty}^{\infty} \frac{dz}{(z^2 + b^2 + v^2 t^2)^{1/2}}, \quad (19)
\end{aligned}$$

which diverges.

Only the field angular momentum (17) is constant in time, so as seems to be expected.

It is surprising that the form (18) is time dependent, since it might seem the most basic form of for the field angular momentum. The computation in eq. (18) ignored the magnetic field outside the long solenoid, although this field is not strictly zero, just extremely small. It turns out that while the “return flux” outside the solenoid can be ignored in computations of the field momentum, this is not the case when calculation the field angular momentum where the field momentum density is multiplied by the vector \mathbf{r} . A more careful computation [38] of $\mathbf{L}_{\text{EM}}^{(2)}$ shows that it is the same as $\mathbf{L}_{\text{EM}}^{(1)}$, *i.e.*, constant in time.

Indeed, a general result (Appendix B of [38]) is that for quasistatic systems, $\mathbf{L}_{\text{EM}}^{(1)} = \mathbf{L}_{\text{EM}}^{(2)}$ (for suitably careful evaluation of the latter) but that the form $\mathbf{L}_{\text{EM}}^{(3)}$ differs from these two. Hence, we identify the field angular momentum in the present example with the result of eq. (17).^{16,17}

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¹⁷Even more subtleties arise if one considers the present example in the rest frame of the electron, such that the solenoid is in motion. In this frame the solenoid appears to have an electric dipole moment [39], and the field of the electron exerts a torque on this moment (whereas there is no such torque in the lab frame). This “paradoxical” torque on an object with a magnetic moment that moves in an electric field is associated with the changing “hidden” mechanical angular momentum of the system, as discussed in [35] for a closely related example.

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