Formulas for Transverse Ionization Cooling in Solenoidal Focusing Channels

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Ionization cooling in solenoidal channels, such as that envisioned for the future muon colliders or neutrino factories, is studied. Assuming that the interaction with the ionization material is weak, the evolution of the transverse emittance and angular momentum can be determined analytically. Simple and practical formulas are derived for a general cooling configuration as well as for periodic channels. The prediction of these formulas agrees well with those obtained from simulation codes. The method developed here should be useful to other areas of beam physics involving solenoidal focusing.

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The possibility of muon colliders [1] or neutrino factories [2]has received much attention recently. One of the key elements in these machines is ionization cooling [3,4]in the solenoidal focusing channel. Until recently, the study of ionization cooling has been carried out mainly by simulation codes such as ICOOL [5]and DPGEANT [6]. Here we present an analytic study of ionization cooling, deriving several simple formulas that should be useful for design of the cooling channel.

We consider the beam evolution in the Larmor frame (rotating at one half of the cyclotron frequency). In this frame, the particle motion in the absence of material is similar to that in a quadrupole channel. The phase space distribution of beam particles is parameterized by the beam emittance, beam angular momentum, and envelope functions. These quantities satisfy coupled, first-order differential equations that may be solved numerically. However, we observe that the envelope functions can be solved to a good approximation by neglecting the effects of the ionization material. The remaining equations for the beam emittance and beam angular momentum can then be solved in a simple, closed form. The study reported here was inspired by the recent work of Penn [7], who studied a set of coupled equations for the envelope functions but did not develop analytic solutions.

We consider a muon beam moving through a series of solenoidal magnets for focusing and absorbing materials for cooling. Let s be the coordinate along the axial direction, and $\mathbf{x} = (x, y)$ is the 2D transverse coordinates. Given the solenoidal field $B(s)$ along the axis, the expression of the magnetic field satisfying Maxwell equations approximately is

$$\mathbf{B}(s, \mathbf{x}) = B(s)\mathbf{e}_s - \frac{1}{2}\frac{dB(s)}{ds}\mathbf{x}\,. \tag{1}$$

Here, $\mathbf{e}_s$ is the unit vector in the s direction. Terms containing higher order s derivatives, which are nonlinear in $\mathbf{x}$, are neglected. Using s as the independent variable, the equation of the transverse motion of a muon in the laboratory frame is

$$\frac{d}{ds}p_s\frac{d\mathbf{x}}{ds} = -qB(s)\mathbf{e}_s \times \frac{d\mathbf{x}}{ds} - \frac{q}{2}\frac{dB}{ds}\mathbf{e}_s \times \mathbf{x} - \eta p_s\frac{d\mathbf{x}}{ds} + p_s\mathbf{u}\,. \tag{2}$$

Here p_s is the momentum in the s direction, q is the muon's charge, $\mathbf{u}$ is the stochastic part of the force due to multiple scattering in material, and

$$\eta = \frac{1}{p_s v}\frac{dE}{ds}\bigg|_{\text{loss}} = \frac{1}{p_s}\frac{dp}{ds}\bigg|_{\text{loss}}\,. \tag{3}$$

Here, $dE/ds|_{\text{loss}}$ is a positive quantity representing the energy loss per unit length in the absorber material, $dp/ds|_{\text{loss}}$ is the corresponding loss in the magnitude of the muon's momentum p, and v is the muon's speed.

It is well known that the precession term $\mathbf{e}_s \times d\mathbf{x}/ds$ in Eq. (2) disappears and the motion becomes harmonic in a certain rotating frame [8]. It is instructive to present a simple derivation of this fact as follows: Consider a coordinate system rotating clockwise around the s axis at an arbitrary rotation rate $\kappa(s)$. The coordinate vector $\mathbf{x}$ in the laboratory frame is related to its rotating frame representation $\mathbf{x}_R$ via

$$\mathbf{x} = R\mathbf{x}_R, \qquad R = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \tag{4}$$

where $\phi(s) = \int_0^s \kappa(\bar{s})\,d\bar{s}$. Differentiating both sides of Eq. (4), we obtain the familiar result in mechanics [9]that the laboratory-frame derivative is a sum of the rotating-frame derivative and the coordinate rotation:

$$\frac{d\mathbf{x}}{ds} = R\left(\frac{d}{ds} - \kappa\mathbf{e}_s\times\right)\mathbf{x}_R\,. \tag{5}$$

By differentiating once more, we also obtain

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$$\frac{d}{ds} p_s \frac{d\mathbf{x}}{ds} = R\left(\frac{d}{ds} - \kappa \mathbf{e}_s \times\right) p_s \left(\frac{d}{ds} - \kappa \mathbf{e}_s \times\right) \mathbf{x}_R . \tag{6}$$

We insert these results into Eq. (2) and try to choose κ so that the precession term $\mathbf{e}_s \times d\mathbf{x}_R/ds$ on both sides of the equation cancel out. The correct choice turns out to be

$$\kappa(s) = \frac{qB(s)}{2p_s}, \tag{7}$$

which is *one-half* of the rotation rate of the precession in the solenoidal field. The factor of one-half arises from the fact that the precession term appears with a factor of two when the right-hand side (rhs) of Eq. (6) is expanded. It is remarkable that the cancellation also occurs for terms containing $dB(s)/ds$, and the equation of motion in this so-called Larmor frame becomes

$$\frac{d}{ds} p_s \frac{d\mathbf{x}_R}{ds} = -p_s \kappa^2 \mathbf{x}_R - p_s \eta \left(\frac{d\mathbf{x}_R}{ds} - \kappa \mathbf{e}_s \times \mathbf{x}_R\right) + p_s \mathbf{u}_R . \tag{8}$$

Here $\mathbf{u}_R = R^{-1}\mathbf{u}$. Note that Eq. (8) is *exact* as long as the magnetic field is given by Eq. (1).

For simplicity, we will henceforth limit this discussion to the case where p_s is constant. Equation (8) can be represented by the following two first-order equations for phase space variables $\mathbf{x}_R$ and $\mathbf{x}'_R$:

$$\frac{d\mathbf{x}_R}{ds} = \mathbf{x}'_R , \qquad \frac{d\mathbf{x}'_R}{ds} = -\kappa^2 \mathbf{x}_R - \eta(\mathbf{x}'_R - \kappa \mathbf{e}_s \times \mathbf{x}_R) + \mathbf{u}_R . \tag{9}$$

Let us now consider the second-order moments that are usually the most important quantities characterizing the beam distribution. If the beam is cylindrically symmetric, there are only four independent second-order moments: $\langle \mathbf{x}_R^2 \rangle$, $\langle \mathbf{x}_R \cdot \mathbf{x}'_R \rangle$, $\langle \mathbf{x}'^2_R \rangle$, and $L = \mathbf{e}_s \cdot \langle \mathbf{x}_R \times \mathbf{x}'_R \rangle = \langle x_R y'_R - y_R x'_R \rangle$. Note that these quantities can also be written as $\langle \mathbf{x}_R^2 \rangle = 2\langle x_R^2 \rangle = 2\langle y_R^2 \rangle$, $\langle \mathbf{x}_R \cdot \mathbf{x}'_R \rangle = 2\langle x_R x'_R \rangle = 2\langle y_R y'_R \rangle$, and $\langle \mathbf{x}'^2_R \rangle = 2\langle x'^2_R \rangle = 2\langle y'^2_R \rangle$. The moments in the rotating frame are related to those in the laboratory frame as follows:

$$\begin{aligned} \langle \mathbf{x}_R^2 \rangle &= \langle \mathbf{x}^2 \rangle , \\ \langle \mathbf{x}_R \cdot \mathbf{x}'_R \rangle &= \langle \mathbf{x} \cdot \mathbf{x}' \rangle , \\ \langle \mathbf{x}'^2_R \rangle &= \langle \mathbf{x}'^2 \rangle - \kappa^2 \langle \mathbf{x}^2 \rangle , \\ L &= \mathbf{e}_s \cdot \langle \mathbf{x} \times \mathbf{x}' \rangle + \kappa \langle \mathbf{x}^2 \rangle . \end{aligned} \tag{10}$$

Note, in particular, that the quantity L may be therefore interpreted as the canonical angular momentum of the beam (divided by p_s). The evolution of the beam moments in the laboratory frame was previously considered [10,11].

The equation for the moments in the Larmor frame follows from Eq. (9):

$$\begin{aligned} \frac{d\langle \mathbf{x}_R^2 \rangle}{ds} &= 2\langle \mathbf{x}_R \cdot \mathbf{x}'_R \rangle , \\ \frac{d\langle \mathbf{x}_R \cdot \mathbf{x}'_R \rangle}{ds} &= \langle \mathbf{x}'^2_R \rangle - \kappa^2 \langle \mathbf{x}_R^2 \rangle - \eta \langle \mathbf{x}_R \cdot \mathbf{x}'_R \rangle , \\ \frac{d\langle \mathbf{x}'^2_R \rangle}{ds} &= -2\kappa^2 \langle \mathbf{x}_R \cdot \mathbf{x}'_R \rangle + 2\kappa\eta L - 2\eta \langle \mathbf{x}'^2_R \rangle + \chi , \\ \frac{dL}{ds} &= -\eta L + \kappa\eta \langle \mathbf{x}_R^2 \rangle . \end{aligned} \tag{11}$$

In the above, χ is the angular excitation due to the stochastic kick $\mathbf{u}$ arising from multiple scattering (MS):

$$\chi(s) = n(s) \langle |\mathbf{u}|^2 \rangle_{\mathrm{MS}} , \tag{12}$$

where $n(s)$ is the average number of multiple scattering events per unit material thickness at s. An approximate formula is [12,13]

$$\chi(s) = \left(\frac{13.6\ \mathrm{MeV}}{pv}\right)^2 \frac{1}{L_{\mathrm{rad}}} , \tag{13}$$

where L_{rad} is the radiation length of the material. Note that the $\chi(s)$ evaluated in the rotating frame is the same as that evaluated in the laboratory frame due to the cylindrical symmetry.

Since Eq. (9) in the absence of material is of the same form as the beam dynamics of quadrupole channels [14], we may introduce the emittance and the envelope functions similarly. Thus, the emittance is defined as follows:

$$\epsilon = \sqrt{\langle \mathbf{x}_R^2 \rangle \langle \mathbf{x}'^2_R \rangle - \langle \mathbf{x}_R \cdot \mathbf{x}'_R \rangle^2} . \tag{14}$$

Note that this is the sum of the usual x and y emittances and thus $\epsilon = 2\epsilon_x$. The beam envelope functions β, α, and γ are also defined in the usual way:

$$\beta = \langle \mathbf{x}_R^2 \rangle / \epsilon , \qquad \alpha = -\langle \mathbf{x}_R \cdot \mathbf{x}'_R \rangle / \epsilon , \qquad \gamma = \langle \mathbf{x}'^2_R \rangle / \epsilon . \tag{15}$$

These quantities, which have been a standard tool in accelerator physics since first introduced by Courant and Snyder [14], provide a convenient representation of the beam envelope containing individual particle trajectories. It follows from Eqs. (14) and Eq. (15) that the envelope functions satisfy the usual relation:

$$\beta\gamma - \alpha^2 = 1 . \tag{16}$$

Equation (11) for the four moments can be converted into an equivalent set involving ϵ, L, α, and β. The equations for ϵ and L can conveniently be written in a matrix form as follows:

$$\frac{d}{ds}\begin{pmatrix} \epsilon \\ L \end{pmatrix} = -\hat{M}(s) \begin{pmatrix} \epsilon \\ L \end{pmatrix} + \begin{pmatrix} \beta\chi \\ 0 \end{pmatrix}, \tag{17}$$

where

$$\hat{M}(s) = \begin{pmatrix} \eta & -\eta\kappa\beta \\ -\eta\kappa\beta & \eta \end{pmatrix}. \tag{18}$$

The equations for the envelope functions are

$$\frac{d\beta}{ds} + 2\alpha = \eta\beta - \frac{\eta\kappa L + \chi}{\epsilon}\beta^2,$$
$$\frac{d\alpha}{ds} + \frac{1+\alpha^2}{\beta} - \kappa^2\beta = -\frac{\eta\kappa L + \chi}{\epsilon}\beta\alpha\,. \tag{19}$$

We now make the crucial observation that Eq. (19) can be solved to a good approximation by neglecting the terms on the rhs. This is because these terms represent the interaction of the beam with materials; the terms containing η for damping and those containing χ for multiple scattering. With this approximation, the function $\beta(s)$ is then determined from the following nonlinear differential equation with suitable boundary conditions, as is familiar in the beam dynamics of quadrupole channels [14]:

$$\frac{1}{2}\frac{d^2\beta}{ds^2} + \kappa^2\beta - \frac{1}{\beta} - \frac{1}{4\beta}\left(\frac{d\beta}{ds}\right)^2 = 0\,. \tag{20}$$

Having determined the function $\beta(s)$, we return to Eq. (17) and find that it can be easily solved in a closed form since the matrices $\hat{M}(s)$ and $\hat{M}(s')$ commute for all s and s'. Introducing the notation,

$$\Psi(s) = \begin{pmatrix}\epsilon(s)\\ L(s)\end{pmatrix} \quad\text{and}\quad \Xi(s) = \begin{pmatrix}\beta(s)\chi(s)\\ 0\end{pmatrix}, \tag{21}$$

the general solution of Eq. (17) is as follows:

$$\Psi(s) = e^{-\hat{\Gamma}(s)}\Psi(0) + e^{-\hat{\Gamma}(s)}\int_0^s d\bar{s}\, e^{\hat{\Gamma}(\bar{s})}\Xi(\bar{s})\,, \tag{22}$$

where

$$\hat{\Gamma}(s) \equiv \int_0^s d\bar{s}\,\hat{M}(\bar{s}) = \begin{pmatrix}\zeta_1(s) & -\zeta_2(s)\\ -\zeta_2(s) & \zeta_1(s)\end{pmatrix}.$$

Here $\zeta_1(s) \equiv \int_0^s d\bar{s}\,\eta(\bar{s})$, $\zeta_2(s) \equiv \int_0^s d\bar{s}\,\eta(\bar{s})\kappa(\bar{s})\beta(\bar{s})$.

Note that our parametrization, which is somewhat different from the one adopted by Penn [7], leads to the intuitively satisfying picture that the quantities ϵ and L are the *beam* properties, while the envelope function β is the property of the focusing channel.

To write Eq. (22) more explicitly, let $\epsilon^0 = \epsilon(0)$ and $L^0 = L(0)$ be the initial values, introduce the quantity

$$\zeta_\pm(s) \equiv \zeta_1 \pm \zeta_2 = \int_0^s d\bar{s}\,\eta(\bar{s})[1 \pm \kappa(\bar{s})\beta(\bar{s})], \tag{23}$$

and note the following identity:

$$e^{\hat{\Gamma}(s)} = \frac{e^{-\zeta_-(s)}}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + \frac{e^{-\zeta_+(s)}}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}. \tag{24}$$

We obtain

$$\epsilon(s) = \frac{e^{-\zeta_-(s)}}{2}[\epsilon^0 + L^0 + \mathcal{D}^-(s)] + \frac{e^{-\zeta_+(s)}}{2}[\epsilon^0 - L^0 + \mathcal{D}^+(s)], \tag{25}$$

$$L(s) = \frac{e^{-\zeta_-(s)}}{2}[\epsilon^0 + L^0 + \mathcal{D}^-(s)] - \frac{e^{-\zeta_+(s)}}{2}[\epsilon^0 - L^0 + \mathcal{D}^+(s)]. \tag{26}$$

Here,

$$\mathcal{D}^\pm(s) = \int_0^s d\bar{s}\, e^{\zeta_\pm(\bar{s})}\beta(\bar{s})\chi(\bar{s}) \tag{27}$$

is the contribution to the emittance generated in the material.

Consider the special case where the material properties and the focusing strength are constant. Assuming that κ is positive, it follows from Eq. (20) that $\beta\kappa$ is unity and ζ_- vanishes. Equation (25) then contains linearly increasing terms for large s coming from $\mathcal{D}^-$; $\epsilon(s) \to (\beta\chi/2)s$ and $L(s) \to (\beta\chi/2)s$. This situation is clearly not desirable.

More efficient cooling can be achieved in a periodic channel in which the sign of κ changes periodically [15]. Thus, we are led to consider a periodic focusing channel in which the beam envelope function $\beta(s)$ is periodic with a period length λ. In this case, by iterating Eq. (22) one obtains

$$\Psi(m\lambda) = e^{-m\hat{\Gamma}(\lambda)}\Psi(0) + \sum_{n=1}^{m} e^{-n\hat{\Gamma}(\lambda)}\int_0^s d\bar{s}\, e^{\hat{\Gamma}(\bar{s})}\Xi(\bar{s})\,. \tag{28}$$

Assuming that neither $\zeta_+(\lambda)$ nor $\zeta_-(\lambda)$ vanishes, the emittance and angular momentum approach well-defined limits as $m \to \infty$ given by

$$\Psi^\infty = \begin{pmatrix}\epsilon^\infty\\ L^\infty\end{pmatrix} = \frac{e^{-\hat{\Gamma}(\lambda)}}{1 - e^{-\hat{\Gamma}(\lambda)}}\int_0^\lambda d\bar{s}\, e^{\hat{\Gamma}(\bar{s})}\Xi(\bar{s})\,. \tag{29}$$

Explicitly,

$$\epsilon^\infty = \frac{1}{2}\left(\frac{e^{-\zeta_-(\lambda)}}{1 - e^{-\zeta_-(\lambda)}}\mathcal{D}^-(\lambda) + \frac{e^{-\zeta_+(\lambda)}}{1 - e^{-\zeta_+(\lambda)}}\mathcal{D}^+(\lambda)\right), \tag{30}$$

$$L^\infty = \frac{1}{2}\left(\frac{e^{-\zeta_-(\lambda)}}{1 - e^{-\zeta_-(\lambda)}}\mathcal{D}^-(\lambda) - \frac{e^{-\zeta_+(\lambda)}}{1 - e^{-\zeta_+(\lambda)}}\mathcal{D}^+(\lambda)\right). \tag{31}$$

The explicit expressions for the emittance and angular momentum at the mth period are

$$\epsilon(m\lambda) = \epsilon^\infty + \frac{e^{-m\zeta_-(\lambda)}}{2}(\epsilon^0 - \epsilon^\infty + L^0 - L^\infty) + \frac{e^{-m\zeta_+(\lambda)}}{2}(\epsilon^0 - \epsilon^\infty - L^0 + L^\infty), \tag{32}$$

$$L(m\lambda) = L^\infty + \frac{e^{-m\zeta_-(\lambda)}}{2}(\epsilon^0 - \epsilon^\infty + L^0 - L^\infty) - \frac{e^{-m\zeta_+(\lambda)}}{2}(\epsilon^0 - \epsilon^\infty - L^0 + L^\infty). \tag{33}$$

There are two damping rates $\zeta_-(\lambda)$ and $\zeta_+(\lambda)$. For efficient damping, these quantities should be designed to be positive and large.

Since usually $\zeta_\pm \ll 1$, Eq. (30) can be further simplified as follows:

$$\epsilon^\infty \simeq \frac{\zeta_1(\lambda)}{\zeta_1(\lambda)^2 - \zeta_2(\lambda)^2}\int_0^\lambda d\bar{s}\,\beta(\bar{s})\chi(\bar{s})\,, \tag{34}$$

TABLE I. Comparison of transverse emittance (cooling factor) calculations for muon cooling channels.

	3.4T FOFO (1)	3.4T FOFO (2)	DFOFO
Eq. (37)	0.19	0.27	0.080
Eq. (25)	0.21	0.34	0.095
ICOOL	0.22	0.34	0.10
DPGEANT	0.24	0.37	N/A

$$L^\infty \simeq \frac{\zeta_2(\lambda)}{\zeta_1(\lambda)^2 - \zeta_2(\lambda)^2} \int_0^\lambda d\bar{s}\, \beta(\bar{s})\chi(\bar{s})\,. \tag{35}$$

Obviously, the condition

$$\zeta_2(\lambda) = \int_0^\lambda \eta(\bar{s})\kappa(\bar{s})\beta(\bar{s})\, d\bar{s} = 0 \tag{36}$$

is necessary to minimize both the emittance and angular momentum. Under this condition, the equilibrium emittance becomes

$$\epsilon^\infty \simeq \frac{\int_0^\lambda d\bar{s}\, \beta(\bar{s})\chi(\bar{s})}{\int_0^\lambda d\bar{s}\, \eta(\bar{s})}\,. \tag{37}$$

The above analysis needs to be modified when either $\zeta_+(\lambda)$ or $\zeta_-(\lambda)$ vanishes. In that case, it is easy to show that $\Psi(m\lambda)$ grows linearly as $m \to \infty$.

The accuracy of the formulas derived here was tested with simulation results using ICOOL and DPGEANT for two cooling channel configurations for the neutrino factory—the 129m FOFO channel reported by Lebrun *et al.* [16] and the latest DFOFO channel proposed by Kim [17]. Table I shows the cooling factors (the ratio of the beginning to the final transverse emittances) calculated with various methods. For the 3.4T FOFO channel, there are two data sets: (i) with a 50-μm Al window and (ii) with a 500-μm Al window. The agreement between our theory and the simulations is remarkable, indicating that the effects of linear beam dynamics dominate the equilibrium beam properties.

Figure 1 shows the transverse emittances calculated from ICOOL, DPGEANT, and Eq. (25) for the FOFO channel with 50 μm Al windows. Clearly, our theory predicts the simulation results [16]very well, and therefore provides an independent check of the simulations. However, the simulation predicts somewhat less cooling performance probably due to the fact that it included nonlinearities and longitudinal dynamics.

In conclusion, we have developed a simple theory for the transverse beam dynamics of ionization cooling in solenoidal focusing channels. The excellent agreement of the predictions of these formulas with simulation results indicates that the theory provides a good understanding of the basic transverse beam dynamics and that the formulas can be used reliably for designing and optimizing various ionization cooling configurations.

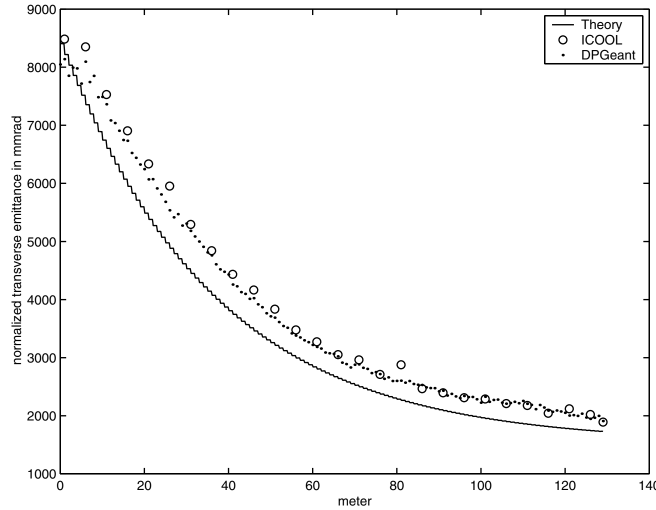

FIG. 1. Transverse cooling calculated with Eq. (25), ICOOL, and DPGEANT. (Simulation data courtesy of P. Lebrun.)

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