The physics of gauge transformations

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A novel approach to classical electromagnetic gauge transformations with an emphasis on the propagation of potentials at finite speeds from charge and current densities is discussed. We consider a new velocity gauge, where the scalar potential propagates at any chosen speed, and the vector potential has a component that propagates at the same speed and another that travels at the finite speed c. Our emphasis on the propagation of the potentials is different from the vector field decomposition of Helmholtz. We discuss two defects of the Helmholtz method and show that our approach corrects them. Our approach enables us to visualize how the potentials propagate in space and see the link between the violation of gauge invariance and the violation of causality and relativity.

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I. INTRODUCTION

A recent paper discusses a novel approach to gauge transformations in classical electrodynamics with an emphasis on the propagation of the potentials at finite speeds from charge and current densities.[1] This dynamical property of the potentials is not discussed in textbooks on electromagnetism and in the literature on gauge transformations in quantum mechanics. This paper is a review of this approach.

The topic of electromagnetic potentials and gauge transformations has a long history. A fascinating, exhaustive account going back to the mid 1800s has been given by Jackson and Okun.[2] As a consequence, the familiar Lorentz gauge in current textbooks will soon be replaced by the Lorenz gauge.[1–3] For the same reasons, the author believes that it is necessary to rename the new class of gauges. We will use the term velocity gauge to unify the complete α-Lorentz gauge, the α-Lorentz gauge, and the alpha-Lorentz gauge used by the author and others,[4–8] and the velocity gauge and the v-gauge used by Jackson[1] and Drury.[9] The reader is encouraged to consult Refs. 1 and 2 for the history as well as the physics of gauge transformations in electromagnetism.

We must discuss the existing way we look at vector potentials. Its central theme is the theorem[3,10–12] due to Helmholtz that any vector field, for example, a vector potential at a fixed time, can be uniquely decomposed into a gradient and a curl component. Most importantly, the curl component has a vanishing divergence and the gradient component does not generate a magnetic field. The gradient component is believed to be totally arbitrary and unnecessary because it can be altered by a gauge transformation. In contrast, the curl component remains unchanged during a gauge transformation. Thus, the Coulomb-gauge vector potential is believed to have only the minimum core (curl) component without any unnecessary gradient components.

In contrast, our approach concentrates on how the potentials propagate in space. We recall that the foremost result of Maxwell's equations is the propagation of electromagnetic fields at the finite speed c from the charge and current densities. We want to understand the potentials in the same way that we have understood the fields. The emphasis is on how each component of the potentials propagates in space, subject to the constraint that the resulting fields always propagate at speed c.

Brill and Goodman[13] were the first to consider the propagation speeds of the potentials and tried to reconcile the finite propagation speed c of the fields with the instantaneous propagation of the Coulomb-gauge scalar potential. Their work was incorporated into Jackson's text[12,3] on classical electrodynamics. At the same time, the author proposed the velocity gauge (the α-Lorentz gauge),[4] in an effort to solve the gauge problem in quantum mechanics.[14] Yang and Kobe[5] used the faster-than-c potentials to establish a link between a violation of gauge invariance and a violation of causality in quantum mechanics. Brown and Crothers[6] developed a comprehensive theoretical framework for the vector potential, with solutions applicable to boundary surfaces. Baxter[7] used this development to re-examine the quantization of the fields. Finally, Drury[9] and Jackson[1] rediscovered the velocity gauge and the v-gauge and were the first to use the term velocity gauge for this gauge.

The organization of this paper is as follows: We devote Sec. II to the existing method by reviewing the necessary mathematics in Sec. II A, the Coulomb and the Lorenz gauges in Sec. II B, and the Helmholtz theorem in Sec. II C. We will show that the Helmholtz decomposition leads to spurious nonlocal and propagation behavior into the gradient and curl components.[3,4,12,13] The discussion in Sec. III parallels the chronology of our approach, with the work of Brill and Goodman[13] on the Coulomb gauge in Sec. III A, the velocity gauge in Sec. III B, the restricted and the unrestricted gauge transformations in Secs. III C and III D, the Brown–Crothers equation in Sec. III E, an alternative derivation of their results in Sec. III F, and Jackson's method in Sec. III G.

Because of the propagation problems of the Coulomb gauge, we review in Sec. IV a new decomposition of the potentials in the Lorenz gauge for a point charge into two sets of c-retarded potentials, such that one set generates only the acceleration fields and the other only the velocity fields.[8] In Sec. V, we discuss the old and new approaches to electromagnetic gauge transformations, discuss how to visualize a gauge by picturing how the potentials propagate in space, and explain how to see the link between the abstract concept of a violation of gauge invariance and the more concrete concept of a violation of causality and relativity. We note that the characteristics and advantages of different gauges in quantum gauge theories have been discussed in Ref. 15.

II. THE MATHEMATICS OF GAUGE TRANSFORMATIONS

We review the traditional discussions of classical electromagnetic potentials and gauge transformations in most textbooks on electromagnetism, add comments where appropriate, and point out some serious defects in the traditional approach.

A. Maxwell's equations and propagation of electromagnetic fields

We first review Maxwell's equations in Gaussian units. Assume that there are localized charge and current densities $\rho(\mathbf{r},t)$ and $\mathbf{J}(\mathbf{r},t)$, subject to the restrictions that they vanish everywhere at $t \leqslant t_0$ and that there are no boundary surfaces. If we use $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ for the electric and magnetic fields, Maxwell's equations in vacuum are

$$\nabla \cdot \mathbf{E}(\mathbf{r},t) = 4\pi\rho(\mathbf{r},t), \tag{2.1a}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0, \tag{2.1b}$$

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t), \tag{2.1c}$$

$$\nabla \times \mathbf{B}(\mathbf{r},t) = \frac{4\pi}{c}\mathbf{J}(\mathbf{r},t) + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t). \tag{2.1d}$$

The wave equations for the fields can be derived from Eq. (2.1):

$$\nabla^2\mathbf{E}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\mathbf{E}(\mathbf{r},t)}{\partial t^2} = 4\pi\nabla\rho(\mathbf{r},t) + \frac{4\pi}{c^2}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t), \tag{2.2a}$$

$$\nabla^2\mathbf{B}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\mathbf{B}(\mathbf{r},t)}{\partial t^2} = -\frac{4\pi}{c}\nabla \times \mathbf{J}(\mathbf{r},t). \tag{2.2b}$$

For our simplified boundary conditions, the closed-form solutions are

$$\mathbf{E}_c(\mathbf{r},t) = -\int G(\mathbf{r},t|c|\mathbf{r}',t')\left[\nabla'\rho(\mathbf{r}',t') + \frac{1}{c^2}\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')\right] \times d\mathbf{r}'dt', \tag{2.3a}$$

$$\mathbf{B}_c(\mathbf{r},t) = \frac{1}{c}\int G(\mathbf{r},t|c|\mathbf{r}',t')[\nabla' \times \mathbf{J}(\mathbf{r}',t')]d\mathbf{r}'dt', \tag{2.3b}$$

where the c-retarded Green function is

$$G(\mathbf{r},t|c|\mathbf{r}',t') = \frac{\delta\left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c} - t'\right)}{|\mathbf{r}-\mathbf{r}'|}, \tag{2.4}$$

where δ is the Dirac delta-function. We have introduced the propagation subscripts c in $\mathbf{E}_c$ and $\mathbf{B}_c$ to emphasize that the fields propagate at speed c from the charge and current densities (the source regions). If a quantity is totally or partially generated outside the source regions, it will not have any propagation subscripts. The propagation subscripts emphasize the most critical dynamical property of the potentials and fields, which we will use extensively in the following.

B. Coulomb gauge, Lorenz gauge, gauge transformations

A different way of obtaining the fields is through the potentials. From Eqs. (2.1b) and (2.1c), it follows that the fields can be generated by a vector potential $\mathbf{A}$ and a scalar potential Φ by

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \mathbf{A}(\mathbf{r},t), \tag{2.5a}$$

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t). \tag{2.5b}$$

By using these results in Eqs. (2.1a) and (2.1d), we obtain

$$\nabla^2\Phi(\mathbf{r},t) + \frac{1}{c}\frac{\partial}{\partial t}[\nabla \cdot \mathbf{A}(\mathbf{r},t)] = -4\pi\rho(\mathbf{r},t), \tag{2.6a}$$

$$\nabla^2\mathbf{A}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\mathbf{A}(\mathbf{r},t)}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}(\mathbf{r},t) + \nabla\left[\nabla \cdot \mathbf{A}(\mathbf{r},t) + \frac{1}{c}\frac{\partial}{\partial t}\Phi(\mathbf{r},t)\right]. \tag{2.6b}$$

We note that Eq. (2.6) does not uniquely determine the potentials because Eq. (2.6) allows the freedom of a gauge transformation. If we use $\chi(\mathbf{r},t)$ to denote a gauge function with the properties, $\nabla\times[\nabla\chi(\mathbf{r},t)]=0$ and $(\partial/\partial t)[\nabla\chi(\mathbf{r},t)] = \nabla[(\partial/\partial t)\chi(\mathbf{r},t)]$, and consider a new set of potentials $(\mathbf{A}',\Phi')$ related to the set $(\mathbf{A},\Phi)$ by a gauge transformation,

$$\mathbf{A}'(\mathbf{r},t) = \mathbf{A}(\mathbf{r},t) + \nabla\chi(\mathbf{r},t),$$
$$\Phi'(\mathbf{r},t) = \Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial}{\partial t}\chi(\mathbf{r},t), \tag{2.7}$$

then it is easy to show that the new potentials also generate the same fields $\mathbf{E}$ and $\mathbf{B}$. Hence the new potentials $\mathbf{A}'$ and Φ' also are solutions of Eq. (2.6).

Because Eq. (2.6) allows multiple solutions, a constraint is needed to solve for a unique set of potentials. Such a constraint is called a gauge condition. The two well-known gauge conditions are the Coulomb and the Lorenz conditions:

$$\nabla \cdot \mathbf{A}^{(C)}(\mathbf{r},t) = 0, \tag{2.8}$$

$$\nabla \cdot \mathbf{A}^{(L)}(\mathbf{r},t) + \frac{1}{c}\frac{\partial}{\partial t}\Phi^{(L)}(\mathbf{r},t) = 0. \tag{2.9}$$

We have introduced the gauge superscripts C and L to distinguish the potentials.[16]

The Lorenz condition in Eq. (2.9) leads to the usual decoupled equations:

$$\nabla^2\Phi^{(L)}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\Phi^{(L)}(\mathbf{r},t)}{\partial t^2} = -4\pi\rho(\mathbf{r},t), \tag{2.10a}$$

$$\nabla^2\mathbf{A}^{(L)}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\mathbf{A}^{(L)}(\mathbf{r},t)}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}(\mathbf{r},t). \tag{2.10b}$$

The closed-form solutions are

$$\Phi_c^{(L)}(\mathbf{r},t) = \int G(\mathbf{r},t|c|\mathbf{r}',t')\rho(\mathbf{r}',t')d\mathbf{r}'dt', \tag{2.11a}$$

$$\mathbf{A}_c^{(L)}(\mathbf{r},t) = \frac{1}{c}\int G(\mathbf{r},t|c|\mathbf{r}',t')\mathbf{J}(\mathbf{r}',t')d\mathbf{r}'dt'. \tag{2.11b}$$

The potentials are subject to the initial and boundary conditions: $\Phi(\mathbf{r},t)=\mathbf{A}(\mathbf{r},t)=0$ for all $\mathbf{r}$ at $t\leqslant t_0$, and $\Phi(\mathbf{r},t)=\mathbf{A}(\mathbf{r},t)=0$, for all t at $|\mathbf{r}|\to\infty$.

The equations for the potentials in the Coulomb gauge can be obtained similarly by using Eq. (2.8) in Eq. (2.6):

$$\nabla^2\Phi^{(C)}(\mathbf{r},t) = -4\pi\rho(\mathbf{r},t), \tag{2.12a}$$

$$\nabla^2\mathbf{A}^{(C)}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2\mathbf{A}^{(C)}(\mathbf{r},t)}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}_{\mathrm{tr}}(\mathbf{r},t), \tag{2.12b}$$

where the transverse current density is

$$\mathbf{J}_{\mathrm{tr}}(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) - \frac{1}{4\pi}\nabla\frac{\partial}{\partial t}\Phi^{(C)}(\mathbf{r},t). \tag{2.13}$$

The solutions, subject to our initial and boundary conditions, are

$$\Phi_\infty^{(C)}(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' = \int \frac{\delta\left(t-\frac{|\mathbf{r}-\mathbf{r}'|}{\infty}-t'\right)}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}',t')d\mathbf{r}'dt', \tag{2.14a}$$

$$\mathbf{A}^{(C)}(\mathbf{r},t) = \frac{1}{c}\int G(\mathbf{r},t|c|\mathbf{r}',t')\mathbf{J}_{\mathrm{tr}}(\mathbf{r}',t')d\mathbf{r}'dt'. \tag{2.14b}$$

In the last term of Eq. (2.14a), we have used the mathematically incorrect but physically intuitive notation of using the infinity sign inside the delta-function to emphasize the physics involved: instantaneity equals propagation at infinitely fast speed.

Here, we note some peculiarities with the potentials in the Coulomb gauge. The scalar potential $\Phi_\infty^{(C)}$ has the subscript ∞ to indicate that it propagates instantaneously from the charge density $\rho(\mathbf{r}',t')$. But, the vector potential does not have a subscript to indicate how it propagates from the source regions. Equation (2.14b) indicates that $\mathbf{A}^{(C)}$ propagates at finite speed c from the transverse current density. But because $\mathbf{J}_{\mathrm{tr}}$ does not vanish outside the source regions because of the last term in Eq. (2.13), the propagation behavior of $\mathbf{A}^{(C)}$ from the source regions cannot easily be determined from Eq. (2.14b). Brill and Goodman[13] were the first to notice these peculiarities and offer a satisfactory explanation (see Sec. III A).

C. Helmholtz theorem

From the Helmholtz theorem, the vector potentials can be uniquely decomposed into a gradient and a curl component (see, for example, Refs. 3 and 10–12). The gradient component is called longitudinal and the curl component is called transverse. With the exception of the transverse current density, we simply refer to them as gradient and curl components. Our reasons will become clear in Sec. IV.

The Helmholtz theorem starts with the identity for a vector function $\mathbf{H}(\mathbf{r},t)$ that vanishes at infinity,

$$\nabla^2\mathbf{H}(\mathbf{r},t) = \nabla[\nabla\cdot\mathbf{H}(\mathbf{r},t)] - \nabla\times[\nabla\times\mathbf{H}(\mathbf{r},t)], \tag{2.15}$$

which can be solved by the usual quasistatic (instantaneous) Green function method:

$$\mathbf{H}(\mathbf{r},t) = \mathbf{H}_{\mathrm{grad}}(\mathbf{r},t) + \mathbf{H}_{\mathrm{curl}}(\mathbf{r},t), \tag{2.16}$$

$$\mathbf{H}_{\mathrm{grad}}(\mathbf{r},t) = -\frac{1}{4\pi}\nabla\int\frac{\nabla'\cdot\mathbf{H}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}', \tag{2.17a}$$

$$\mathbf{H}_{\mathrm{curl}}(\mathbf{r},t) = \frac{1}{4\pi}\nabla\times\int\frac{\nabla'\times\mathbf{H}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' = \mathbf{H}(\mathbf{r},t) - \mathbf{H}_{\mathrm{grad}}(\mathbf{r},t). \tag{2.17b}$$

There are two physics-related problems with this decomposition that are relevant here: It introduces a spurious nonlocal property[3,4,12,13] and spurious propagation behavior into the gradient and curl components. The results of the Helmholtz decomposition are not physically consistent with the original vector function because of the spurious properties of its components.

We first examine the spurious nonlocal property by considering the Helmholtz decomposition of the current density $\mathbf{J}(\mathbf{r},t)$ from Eq. (2.1d). According to Eq. (2.17a), the gradient component of the current density is (see, for example, Ref. 3)

$$\mathbf{J}_{\mathrm{grad}}(\mathbf{r},t) = -\frac{1}{4\pi}\nabla\int\frac{\nabla'\cdot\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' = \frac{1}{4\pi}\nabla\frac{\partial}{\partial t}\int\frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' = \frac{1}{4\pi}\nabla\frac{\partial}{\partial t}\Phi^{(C)}(\mathbf{r},t), \tag{2.18}$$

where we have used the equation of continuity of the charge and current densities and $\Phi^{(C)}$ is the Coulomb potential in Eq. (2.14a). Because $\mathbf{J}_{\mathrm{curl}}=\mathbf{J}-\mathbf{J}_{\mathrm{grad}}$, we see that $\mathbf{J}_{\mathrm{curl}}$ is the transverse current density $\mathbf{J}_{\mathrm{tr}}$ in (2.13). Because the Coulomb potential $\Phi^{(C)}$ exists everywhere, it is obvious that both $\mathbf{J}_{\mathrm{grad}}$ and $\mathbf{J}_{\mathrm{curl}}$ do not in general vanish outside the source regions. Hence, both $\mathbf{J}_{\mathrm{grad}}$ and $\mathbf{J}_{\mathrm{curl}}$ cannot be physically measured.

Next, we discuss the spurious propagation behavior. We assume that the sources are turned on at $t=0$ and are turned off at t_f and consider a time τ satisfying $0<\tau<t_f$. Because the sources are localized, there exists a finite, positive R_s such that $\rho(\mathbf{r},t)=\mathbf{J}(\mathbf{r},t)=0$ for all $\mathbf{r}$ and t satisfying $|\mathbf{r}|>R_s$ and $0\leqslant t\leqslant\tau$. Because the potentials in the Lorenz gauge propagate at speed c from the charge and current densities, we have $\Phi_c^{(L)}(\mathbf{r},\tau)=\mathbf{A}_c^{(L)}(\mathbf{r},\tau)=0$ for all $\mathbf{r}$ with $|\mathbf{r}|>(R_s+c\tau)$. Because the sources are on from $t=0$ to $t=\tau$ and beyond, there are regions with $|\mathbf{r}|<(R_s+c\tau)$ where either the vector or the scalar potential or both do not vanish.

We now consider the Helmholtz decomposition of the vector potential in the Lorenz gauge at $t=\tau$,

$$[\mathbf{A}_c^{(L)}]_{\mathrm{grad}}(\mathbf{r},\tau) = -\frac{1}{4\pi}\nabla\int\frac{\nabla'\cdot\mathbf{A}_c^{(L)}(\mathbf{r}',\tau)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' = \frac{1}{4\pi c}\nabla\frac{\partial}{\partial\tau}\int\frac{\Phi_c^{(L)}(\mathbf{r}',\tau)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}', \tag{2.19a}$$

$$[\mathbf{A}_c^{(L)}]_{\text{curl}}(\mathbf{r},\tau)=\mathbf{A}_c^{(L)}(\mathbf{r},\tau)-[\mathbf{A}_c^{(L)}]_{\text{grad}}(\mathbf{r},\tau). \tag{2.19b}$$

We concentrate on the regions with $|\mathbf{r}|>(R_s+c\tau)$. According to Eq. (2.19a), the gradient component $[\mathbf{A}_c^{(L)}]_{\text{grad}}(\mathbf{r},\tau)$ does not vanish because the scalar potential does not vanish identically for all $\mathbf{r}'$ satisfying $|\mathbf{r}'|\leqslant(R_s+c\tau)$, and the integration $d\mathbf{r}'$ is over the entire space, including all $|\mathbf{r}'|\leqslant(R_s+c\tau)$. Most importantly, the connection between the source $\Phi_c^{(L)}(\mathbf{r}',\tau)$ with $|\mathbf{r}'|\leqslant(R_s+c\tau)$ and the destination $[\mathbf{A}_c^{(L)}]_{\text{grad}}(\mathbf{r},\tau)$ with $|\mathbf{r}|>(R_s+c\tau)$ is instantaneous (quasistatic) because the same time τ appears in both. But the vector potential $\mathbf{A}_c^{(L)}(\mathbf{r},\tau)$ does vanish for all $|\mathbf{r}|>(R_s+c\tau)$ because of its finite propagation speed c. These arguments indicate that the gradient component $[\mathbf{A}_c^{(L)}]_{\text{grad}}$ propagates ahead of its progenitor $\mathbf{A}_c^{(L)}$. By Eq. (2.19b), the curl component $[\mathbf{A}_c^{(L)}]_{\text{curl}}$ also possesses the same spurious propagation behavior.

This discussion indicates a need for a physically consistent understanding of the potentials. The emphasis will be on the physics of the potentials: how and how fast the potentials propagate in space. We will see that this new way gives us a physically intuitive way to understand electromagnetic gauge transformations.

III. THE PHYSICS OF GAUGE TRANSFORMATIONS: PROPAGATION OF THE POTENTIALS

Our criticism of the traditional approach centers on its reliance on the mathematics of the gradient and the curl components from the Helmholtz theorem. We will see that the new approach corrects for this important deficiency in the traditional approach.

A. Instantaneous component of the vector potential in the Coulomb gauge

It was well known that the scalar potential in the Coulomb gauge propagates instantaneously in space even before the appearance of Ref. 13. Jackson emphasized this property[10,11] and stated: “The scalar potential is just the instantaneous Coulomb potential due to the charge density $\rho(\mathbf{x},t)$.”

To reconcile this causality-violating behavior of the scalar potential with the finite propagation speed c of the fields, Ref. 13 concluded that the transverse current density must exist outside the source regions, and the Coulomb-gauge vector potential must contain two components with different propagation speeds: one propagates instantaneously while the other propagates at the finite speed c from the charge and current densities.

In Sec. II C we discussed the nonlocal property of the transverse current density. We now focus on the propagation behavior of the potentials. We solve for the vector potential from Eq. (2.5b) and obtain

$$\mathbf{A}^{(C)}(\mathbf{r},t)=-c\int^{t}\mathbf{E}_c(\mathbf{r},t'')dt''-c\int^{t}\nabla\Phi_\infty^{(C)}(\mathbf{r},t'')dt'', \tag{3.1}$$

which clearly shows the propagation behavior of $\mathbf{A}^{(C)}$ first found in Ref. 13.

We still have to reconcile the differences in physics between Eqs. (3.1) and (2.14b). In Eq. (3.1), $\mathbf{A}^{(C)}$ has two propagation speeds, c and ∞, from the charge and current densities. In Eq. (2.14b), $\mathbf{A}^{(C)}$ propagates at only one speed c from the transverse current density. The resolution lies in the nonlocal property of the transverse current density (see Refs. 3, 4, 12, and 13). To see this, we rewrite Eq. (2.14b) as

$$\mathbf{A}^{(C)}(\mathbf{r},t)=\mathbf{A}_c^{(L)}(\mathbf{r},t)+\mathbf{A}^{(C,\Phi)}(\mathbf{r},t), \tag{3.2}$$

where $\mathbf{A}_c^{(L)}$ is the vector potential in the Lorenz gauge in Eq. (2.11b) and

$$\mathbf{A}^{(C,\Phi)}(\mathbf{r},t)=-\frac{1}{4\pi c}\int d\mathbf{r}''dt''G(\mathbf{r},t|c|\mathbf{r}'',t'') \times\left[\frac{\partial}{\partial t''}\nabla''\int\frac{\delta(t''-t')}{|\mathbf{r}''-\mathbf{r}'|}\rho(\mathbf{r}',t')d\mathbf{r}'dt'\right]. \tag{3.3}$$

As is clear from Eq. (3.3), the propagation from $(\mathbf{r}',t')$ to $(\mathbf{r}'',t'')$ is instantaneous, which means that $t''=t'$. The propagation from $(\mathbf{r}'',t'')$ to $(\mathbf{r},t)$ is at finite speed c, which means that $t=t''+|\mathbf{r}-\mathbf{r}''|/c$. Hence, $t=t'+|\mathbf{r}-\mathbf{r}''|/c$. But, we have to integrate $d\mathbf{r}''$ over the entire space, including $\mathbf{r}''=\mathbf{r}$. Thus, the component $\mathbf{A}^{(C,\Phi)}$ has the net effect of instantaneous propagation from the charge density.

That the Coulomb-gauge vector potential has an instantaneous component immediately leads to the conclusion that the instantaneous component is spurious and unnecessary for the generation of the c-retarded fields. This conclusion follows by considering the restriction imposed by relativity that no physical information can propagate faster than c in vacuum. Hence the instantaneous component cannot be related to any physical information, for example, the electric fields.

B. Velocity gauge

As stated in Sec. I, our main purpose is to understand the propagation of the potentials in the same way we understand the propagation of the fields. Hence the emphasis is on how each component of the potentials propagates in space, subject to the constraint that the resulting fields always propagate at speed c.

To understand the effects of gauge transformations in quantum mechanics,[14] the author discovered the velocity gauge (the α-Lorentz gauge).[4] This class of gauges uses a parameter to exhibit the propagation behavior of the potentials; the Coulomb and the Lorenz gauges are just special cases. The velocity gauge with the parameter v, abbreviated as the v-gauge, is defined by

$$\nabla\cdot\mathbf{A}^{(v)}(\mathbf{r},t)+\frac{c^2}{v^2}\frac{\partial}{c\partial t}\Phi^{(v)}(\mathbf{r},t)=0, \tag{3.4}$$

where v is a real, nonzero constant. Our purpose is to find closed-form solutions for the potentials having definite propagation behavior from the charge and current densities.

First, we substitute Eq. (3.4) in Eq. (2.6) to eliminate $\nabla\cdot\mathbf{A}^{(v)}$:

$$\nabla^2\Phi^{(v)}(\mathbf{r},t)-\frac{1}{v^2}\frac{\partial^2\Phi^{(v)}(\mathbf{r},t)}{\partial t^2}=-4\pi\rho(\mathbf{r},t), \tag{3.5a}$$

$$\nabla^2\mathbf{A}^{(v)}(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2\mathbf{A}^{(v)}(\mathbf{r},t)}{\partial t^2}=-\frac{4\pi}{c}\mathbf{J}_{vt}(\mathbf{r},t), \tag{3.5b}$$

where the v-transverse current density is

$$\mathbf{J}_{vt}(\mathbf{r},t)=\mathbf{J}(\mathbf{r},t)+\frac{1}{4\pi}\left(\frac{c^2}{v^2}-1\right)\nabla\frac{\partial}{\partial t}\Phi^{(v)}(\mathbf{r},t). \tag{3.6}$$

We define the v-propagating Green function by

$$G(\mathbf{r},t|v|\mathbf{r}',t')=\frac{\delta\left(t-\frac{|\mathbf{r}-\mathbf{r}'|}{v}-t'\right)}{|\mathbf{r}-\mathbf{r}'|}, \tag{3.7}$$

$$\left(\nabla^2-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r},t|v|\mathbf{r}',t')=-4\pi\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{3.8}$$

The solutions for the potentials are

$$\Phi_v^{(v)}(\mathbf{r},t)=\int G(\mathbf{r},t|v|\mathbf{r}',t')\rho(\mathbf{r}',t')d\mathbf{r}'dt', \tag{3.9a}$$

$$\mathbf{A}^{(v)}(\mathbf{r},t)=\frac{1}{c}\int G(\mathbf{r},t|c|\mathbf{r}',t')\mathbf{J}_{vt}(\mathbf{r}',t')d\mathbf{r}'dt'. \tag{3.9b}$$

Again, the vector potential has no propagation subscripts because of the nonlocal behavior of $\mathbf{J}_{vt}(\mathbf{r}',t')$.

The following argument will be used to obtain the closed-form solution for the vector potential: Electromagnetic fields propagate with speed c from the physical charge and current densities and are gauge-invariant. If we substitute the potentials in Eq. (2.5) and show the propagation speeds of all those quantities that have them, then

$$\mathbf{B}_c(\mathbf{r},t)=\nabla\times\mathbf{A}^{(v)}(\mathbf{r},t), \tag{3.10a}$$

$$\mathbf{E}_c(\mathbf{r},t)=-\nabla\Phi_v^{(v)}(\mathbf{r},t)-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}^{(v)}(\mathbf{r},t). \tag{3.10b}$$

In the following we will consider only those cases with $v\neq c$, including $v<0$ for the advanced solutions. When $v=c$, the solutions are the potentials in the Lorenz gauge.

We note that if Eq. (3.10b) is to be mathematically correct and physically consistent, $\mathbf{A}^{(v)}$ must have one c-retarded and one v-propagating component:

$$\mathbf{A}^{(v)}(\mathbf{r},t)=\mathbf{A}_c^{(v)}(\mathbf{r},t)+\mathbf{A}_v^{(v)}(\mathbf{r},t). \tag{3.11}$$

The v-component must cancel exactly the effects of the v-propagating scalar potential so that only the c-retarded electric field is generated,

$$-\nabla\Phi_v^{(v)}(\mathbf{r},t)-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_v^{(v)}(\mathbf{r},t)=0, \tag{3.12a}$$

$$-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_c^{(v)}(\mathbf{r},t)=\mathbf{E}_c(\mathbf{r},t). \tag{3.12b}$$

If we substitute Eq. (3.11) in Eq. (3.10a) for the generation of the magnetic field, we have

$$\nabla\times\mathbf{A}_v^{(v)}(\mathbf{r},t)=0, \tag{3.13a}$$

$$\nabla\times\mathbf{A}_c^{(v)}(\mathbf{r},t)=\mathbf{B}_c(\mathbf{r},t). \tag{3.13b}$$

The solution of Eqs. (3.12a) and (3.13a) is

$$\mathbf{A}_v^{(v)}(\mathbf{r},t)=-c\nabla\int_{t_0}^{t}\Phi_v^{(v)}(\mathbf{r},t'')dt''. \tag{3.14}$$

To solve for $\mathbf{A}_c^{(v)}$, we use the fact that the fields are gauge-invariant, which means that they also can be expressed in terms of the c-retarded potentials in the Lorenz gauge:

$$-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_c^{(v)}(\mathbf{r},t)=\mathbf{E}_c(\mathbf{r},t)=-\nabla\Phi_c^{(L)}(\mathbf{r},t)-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_c^{(L)}(\mathbf{r},t), \tag{3.15a}$$

$$\nabla\times\mathbf{A}_c^{(v)}(\mathbf{r},t)=\mathbf{B}_c(\mathbf{r},t)=\nabla\times\mathbf{A}_c^{(L)}(\mathbf{r},t). \tag{3.15b}$$

The solution of Eq. (3.15) is

$$\mathbf{A}_c^{(v)}(\mathbf{r},t)=\mathbf{A}_c^{(L)}(\mathbf{r},t)+c\nabla\int_{t_0}^{t}\Phi_c^{(L)}(\mathbf{r},t'')dt''. \tag{3.16}$$

Thus, we obtain the desired closed-form solution for the vector potential:

$$\mathbf{A}^{(v)}(\mathbf{r},t)=\mathbf{A}_c^{(L)}(\mathbf{r},t)+c\nabla\int_{t_0}^{t}[\Phi_c^{(L)}(\mathbf{r},t'')-\Phi_v^{(v)}(\mathbf{r},t'')]dt''. \tag{3.17}$$

The solutions in Eqs. (3.9a) and (3.17) are valid for all values of $v\neq 0$, including $v<0$.

Thus, a choice of gauge determines not only the gradient component of the vector potential, but also the propagation speeds of the potentials. Also, Eqs. (3.12a) and (3.13a) illustrate the essence of gauge invariance: the unphysical or unnecessary components of the potentials cancel exactly, generating no electric and magnetic fields.

For completeness, we show the equivalence of the two expressions in Eqs. (3.9b) and (3.17), along with the result of Ref. 5. We first write Eq. (3.9b) in the form:

$$\mathbf{A}^{(v)}(\mathbf{r},t)=\mathbf{A}_c^{(L)}(\mathbf{r},t)+\frac{1}{4\pi}\left(\frac{c^2}{v^2}-1\right)\nabla\Lambda^{(v,\Phi)}(\mathbf{r},t), \tag{3.18}$$

$$\Lambda^{(v,\Phi)}(\mathbf{r},t)=\frac{\partial}{c\partial t}\int G(\mathbf{r},t|c|\mathbf{r}'',t'')\Phi_v^{(v)}(\mathbf{r}'',t'')d\mathbf{r}''dt''. \tag{3.19}$$

We then differentiate $\Lambda^{(v,\Phi)}$ with respect to time and use Eq. (3.5) to derive an expression for $\partial\Lambda^{(v,\Phi)}/\partial t$ in terms of the potentials $\Phi_v^{(v)}$ and $\Phi_c^{(L)}$:

$$\begin{aligned}\frac{\partial}{c\partial t}\Lambda^{(v,\Phi)}(\mathbf{r},t)&=\int[4\pi\delta(\mathbf{r}-\mathbf{r}'')\delta(t-t'')\\&\quad+\nabla^2G(\mathbf{r},t|c|\mathbf{r}'',t'')]\Phi_v^{(v)}(\mathbf{r}'',t'')d\mathbf{r}''dt''\\&=4\pi\Phi_v^{(v)}(\mathbf{r},t)+\int G(\mathbf{r},t|c|\mathbf{r}'',t'')\\&\quad\times[\nabla''^2\Phi_v^{(v)}(\mathbf{r}'',t'')]d\mathbf{r}''dt''\\&=4\pi\Phi_v^{(v)}(\mathbf{r},t)-4\pi\Phi_c^{(L)}(\mathbf{r},t)\\&\quad+\frac{c^2}{v^2}\frac{\partial}{c\partial t}\Lambda^{(v,\Phi)}(\mathbf{r},t).\end{aligned} \tag{3.20}$$

From Eqs. (3.20) and (3.18), we arrive at Eq. (3.17), which completes our proof.

C. Restricted gauge transformations

There are two kinds of classical electromagnetic gauge transformations. However, we will not call them gauge transformations of the first and second kinds because these terms have been used before.[17] Instead, we adopt Jackson's convention[3,11,12] and classify one kind as the restricted gauge transformations and all others as unrestricted gauge transformations.

We use $\chi^{(v)}$ to denote a solution to the homogeneous v-wave equation,

$$\nabla^2\chi^{(v)}(\mathbf{r},t)-\frac{1}{v^2}\frac{\partial^2\chi^{(v)}(\mathbf{r},t)}{\partial t^2}=0. \tag{3.21}$$

Then, we call the transformation,

$$\mathbf{A}^{(\mathrm{new})}=\mathbf{A}^{(v)}+\nabla\chi^{(v)},\quad \Phi^{(\mathrm{new})}=\Phi^{(v)}-\frac{1}{c}\frac{\partial}{\partial t}\chi^{(v)}, \tag{3.22}$$

a restricted gauge transformation within the v-gauge. It has two unusual properties: First, the new potentials satisfy the same gauge condition as the old potentials,

$$\nabla\cdot\mathbf{A}^{(\mathrm{new})}+\frac{c^2}{v^2}\frac{\partial}{c\partial t}\Phi^{(\mathrm{new})}=\nabla\cdot\mathbf{A}^{(v)}+\frac{c^2}{v^2}\frac{\partial}{c\partial t}\Phi^{(v)}+\nabla^2\chi^{(v)}-\frac{1}{v^2}\frac{\partial^2\chi^{(v)}}{\partial t^2}=0. \tag{3.23}$$

Second, because the v-wave equation (3.21) is homogeneous, this transformation can never alter the components of $\mathbf{A}^{(v)}$ and $\Phi^{(v)}$ which are explicitly related to the charge and current densities, that is, those given in Eqs. (3.9a) and (3.17). Hence, it can never transform the potentials to another v'-gauge if $v'\neq v$.

D. Unrestricted gauge transformations

The transformations that alter the gauge conditions or the propagation speeds belong to the unrestricted gauge transformations. Explicit gauge functions for a broad range of the unrestricted gauge transformations have been obtained (Ref. 4, Appendix B) and, more recently and more extensively in Ref. 1.

We use $\chi^{(\mathrm{old}\rightarrow\mathrm{new})}$ to denote the gauge function that transforms the old potentials $\mathbf{A}^{(\mathrm{old})}$ and $\Phi^{(\mathrm{old})}$ to the new potentials $\mathbf{A}^{(\mathrm{new})}$ and $\Phi^{(\mathrm{new})}$, as described in Eq. (2.7). We also define the potentials $\Phi^{(T)}$ and $\mathbf{A}^{(T)}$ in the temporal gauge (see, for example, Ref. 1) by

$$\Phi^{(T)}(\mathbf{r},t)=0,$$

$$\mathbf{A}_c^{(T)}(\mathbf{r},t)=\mathbf{A}_c^{(L)}(\mathbf{r},t)+c\int^t\nabla\Phi_c^{(L)}(\mathbf{r},t'')dt'', \tag{3.24a}$$

$$\frac{\partial}{c\partial t}[\nabla\cdot\mathbf{A}^{(T)}(\mathbf{r},t)]=-4\pi\rho(\mathbf{r},t). \tag{3.24b}$$

Then, for two arbitrary nonzero v and u,

$$\chi^{(v\rightarrow T)}(\mathbf{r},t)=c\int^t\Phi_v^{(v)}(\mathbf{r},t'')dt'', \tag{3.25}$$

$$\begin{aligned}\chi^{(v\rightarrow u)}(\mathbf{r},t)&=\chi^{(v\rightarrow T)}(\mathbf{r},t)-\chi^{(u\rightarrow T)}(\mathbf{r},t)\\&=c\int^t[\Phi_v^{(v)}(\mathbf{r},t'')-\Phi_u^{(u)}(\mathbf{r},t'')]dt''\\&=c\int d\mathbf{r}'\frac{1}{R}\int_{t-R/u}^{t-R/v}\rho(\mathbf{r}',t')dt',\end{aligned} \tag{3.26}$$

where $R=|\mathbf{r}-\mathbf{r}'|$ and the last expression in Eq. (3.26) is in the form favored in Ref. 1.

E. Brown–Crothers equation

The mathematical groundwork for the velocity gauge was developed in Ref. 6 that noted two deficiencies when the gauge was first introduced.[4] These deficiencies are that Eq. (3.5b) for the vector potential is still coupled to the scalar potential, and the solution for the vector potential in Eq. (3.17) is not applicable to problems with boundary surfaces.

Instead of following their Lagrangian field-theoretic approach, we use Eq. (3.4) to eliminate the scalar potential from Eq. (2.6b), resulting in the Brown–Crothers equation:

$$\nabla^2\mathbf{A}^{(v)}(\mathbf{r},t)-\frac{1}{c^2}\frac{\partial^2\mathbf{A}^{(v)}(\mathbf{r},t)}{\partial t^2}+\left(\frac{v^2}{c^2}-1\right)\nabla[\nabla\cdot\mathbf{A}^{(v)}(\mathbf{r},t)]=-\frac{4\pi}{c}\mathbf{J}(\mathbf{r},t). \tag{3.27}$$

To solve Eq. (3.27), a dyadic Green function $\mathbf{G}^{(v)}\times(\mathbf{r},t\|\mathbf{r}',t')$ is introduced,

$$\mathbf{A}^{(v)}(\mathbf{r},t)=\frac{1}{c}\int\mathbf{G}^{(v)}(\mathbf{r},t\|\mathbf{r}',t')\cdot\mathbf{J}(\mathbf{r}',t')d\mathbf{r}'dt', \tag{3.28}$$

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{G}^{(v)}+\left(\frac{v^2}{c^2}-1\right)\nabla[\nabla\cdot\mathbf{G}^{(v)}]=-4\pi\mathbf{I}\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'), \tag{3.29}$$

where $\mathbf{I}$ is the dyadic idemfactor. Then, Eq. (3.29) is further reduced to a fourth-order equation so that the solution becomes more obvious. First, we take the divergence of both sides of Eq. (3.29) to find

$$\left(\nabla^2-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)[\nabla\cdot\mathbf{G}^{(v)}]=-\frac{4\pi c^2}{v^2}\nabla\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'). \tag{3.30}$$

Then we apply the operator $(\nabla^2-v^{-2}\partial^2/\partial t^2)$ from the left to both sides of Eq. (3.29) and use Eq. (3.30) to obtain

$$\begin{aligned}\left(\nabla^2-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{G}^{(v)}&=\mathbf{I}\left(\nabla^2-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)[(-4\pi)\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t')]\\&\quad+(-4\pi)^2\frac{v^2-c^2}{4\pi v^2}\nabla\nabla\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t').\end{aligned} \tag{3.31}$$

The solution of this fourth-order equation is[18]

$$\mathbf{G}^{(v)}(\mathbf{r},t\|\mathbf{r}',t') = G(\mathbf{r},t|c|\mathbf{r}',t')\mathbf{I} + \frac{v^2-c^2}{4\pi v^2}\nabla\,\nabla\,G(\mathbf{r},t|c|v|\mathbf{r}',t'), \tag{3.32}$$

where

$$G(\mathbf{r},t|c|v|\mathbf{r}',t') = \int G(\mathbf{r},t|c|\mathbf{r}'',t'')G(\mathbf{r}'',t''|v|\mathbf{r}',t')d\mathbf{r}''dt'', \tag{3.33}$$

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G^{(v)}(\mathbf{r},t|c|v|\mathbf{r}',t') = (-4\pi)^2\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{3.34}$$

The solution in Eq. (3.32) also can be expressed in our preferred operator form,

$$\mathbf{G}^{(v)}_{op}(\mathbf{r},t\|\mathbf{r}',t') = G(\mathbf{r},t|c|\mathbf{r}',t')\mathbf{I} + \frac{v^2-c^2}{4\pi v^2}G(\mathbf{r},t|c|v|\mathbf{r}',t')\nabla'\nabla', \tag{3.35}$$

which can replace $\mathbf{G}^{(v)}$ in Eq. (3.28) for the vector potential. Yang and Hirschfelder[19] used the classical electric and magnetic multipole operators extensively in their investigation of the electromagnetic interactions of semirelativistic classical charged particles with electric and magnetic multipole moments.

F. Alternative derivation of results of Brown and Crothers

We now show that the dyadic Green function operator $\mathbf{G}^{(v)}_{\text{op}}$ in Eq. (3.35) or the Green function $\mathbf{G}^{(v)}$ in Eq. (3.32) also can be derived directly from Eq. (3.9b). First, we separate the vector potential in Eq. (3.9b) into two terms:

$$\mathbf{A}^{(v)}(\mathbf{r},t) = \mathbf{A}^{(L)}_c(\mathbf{r},t) + \mathbf{A}^{(v,\Phi)}(\mathbf{r},t), \tag{3.36}$$

where $\mathbf{A}^{(L)}_c$ is the vector potential in the Lorenz gauge in Eq. (2.11b) and

$$\begin{aligned}\mathbf{A}^{(v,\Phi)}(\mathbf{r},t) &= \frac{c^2-v^2}{4\pi c v^2}\int d\mathbf{r}''dt''G(\mathbf{r},t|c|\mathbf{r}'',t'')\nabla''\frac{\partial}{\partial t''}\Phi^{(v)}_v(\mathbf{r}'',t'')\\ &= \frac{c^2-v^2}{4\pi c v^2}\int d\mathbf{r}''dt''G(\mathbf{r},t|c|\mathbf{r}'',t'')\nabla''\frac{\partial}{\partial t''}\\ &\quad\times\int G(\mathbf{r}'',t''|v|\mathbf{r}',t')\rho(\mathbf{r}',t')d\mathbf{r}'dt'\\ &= \frac{c^2-v^2}{4\pi c v^2}\int d\mathbf{r}''dt''G(\mathbf{r},t|c|\mathbf{r}'',t'')\\ &\quad\times\int G(\mathbf{r}'',t''|v|\mathbf{r}',t')\nabla'\left[\frac{\partial}{\partial t'}\rho(\mathbf{r}',t')\right]d\mathbf{r}'dt'\\ &= \frac{1}{c}\frac{v^2-c^2}{4\pi v^2}\int[G(\mathbf{r},t|c|v|\mathbf{r}',t')\nabla'\nabla']\\ &\quad\cdot\mathbf{J}(\mathbf{r}',t')d\mathbf{r}'dt'.\end{aligned} \tag{3.37}$$

Thus we see that we can derive the same dyadic Green function in a more straightforward, albeit less elegant, manner.

G. Jackson's method

There is yet another simple and elegant method by Jackson[1] for obtaining the vector potential in an arbitrary gauge. It requires three known quantities to start with: $\mathbf{A}^{(\text{old})}$, $\Phi^{(\text{old})}$, and $\Phi^{(\text{new})}$, where $\mathbf{A}^{(\text{old})}$ and $\Phi^{(\text{old})}$ are the potentials in a known gauge and $\Phi^{(\text{new})}$ is the scalar potential in a new gauge with an unknown vector potential $\mathbf{A}^{(\text{new})}$. Unlike our method in Sec. III B where the physics of the propagation of the potentials is used extensively, Jackson's method is strictly a mathematical procedure. The reader is advised to consult Ref. 1 for many more gauge functions with historical importance. Drury[9] also rediscovered the velocity gauge, but did not derive the closed-form solutions for the vector potentials.

Let us demonstrate how Jackson's method works. We start with the gauge transformation in Eq. (2.7), solve for the gauge function, and then use this solution to solve for the vector potential:

$$\chi^{(\text{old}\to\text{new})}(\mathbf{r},t) = c\int^t[\Phi^{(\text{old})}(\mathbf{r},t'') - \Phi^{(\text{new})}(\mathbf{r},t'')]dt'', \tag{3.38a}$$

$$\mathbf{A}^{(\text{new})}(\mathbf{r},t) = \mathbf{A}^{(\text{old})}(\mathbf{r},t) + c\,\nabla\int^t[\Phi^{(\text{old})}(\mathbf{r},t'') - \Phi^{(\text{new})}(\mathbf{r},t'')]dt''. \tag{3.38b}$$

As an example, we apply this method to find the vector potentials in the v-gauge from the Lorenz gauge. With $R = |\mathbf{r}-\mathbf{r}'|$, we have

$$\begin{aligned}\chi^{(L\to v)}(\mathbf{r},t) &= c\int^t[\Phi^{(L)}_c(\mathbf{r},t'') - \Phi^{(v)}_v(\mathbf{r},t'')]dt''\\ &= c\int d\mathbf{r}'\int_{t-R/v}^{t-R/c}\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}dt',\end{aligned} \tag{3.39}$$

$$\begin{aligned}\mathbf{A}^{(v)}(\mathbf{r},t) &= \mathbf{A}^{(L)}_c(\mathbf{r},t) + c\,\nabla\int^t[\Phi^{(L)}_c(\mathbf{r},t'') - \Phi^{(v)}_v(\mathbf{r},t'')]dt''\\ &= \mathbf{A}^{(L)}_c(\mathbf{r},t) + c\,\nabla\int d\mathbf{r}'\int_{t-R/v}^{t-R/c}\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}dt'.\end{aligned} \tag{3.40}$$

Just like our method in Sec. III B, Jackson's method relies on the work of Ref. 6 as its mathematical foundation.

IV. POTENTIALS FOR THE VELOCITY AND ACCELERATION FIELDS OF A POINT-CHARGED PARTICLE

In the traditional approach, the vector potential $\mathbf{A}^{(C)}$ in the Coulomb gauge is believed to generate the free fields because its divergence vanishes identically. But $\mathbf{A}^{(C)}$ carries a causality-violating instantaneous component. Hence, not all of the free E-field $\mathbf{E}^{(C)} = -(1/c)(\partial\mathbf{A}^{(C)}/\partial t)$ can be physically measured. Only the c-retarded component of $\mathbf{E}^{(C)}$ can be measured. For this reason, we want to revisit an original objective of using the Coulomb gauge in quantum mechanics: to find a set of c-retarded potentials that generates only the c-retarded, transverse, radiation or free fields.

Instead of trying to solve for such a set of potentials for an arbitrary charge and current distribution, we consider the special case of a point-charged particle. Here, the situation is well defined: the exact expressions for the velocity and the acceleration fields are available,[20–22] and the acceleration fields are the radiation fields. We want to find a set of c-retarded potentials for the acceleration fields of a point-charged particle with an arbitrary motion in any Lorentz frame.

The total fields of a point-charged particle separate naturally into the velocity and the acceleration fields, and both fields are c-retarded. The velocity fields are the bound fields because they vary like R^{-2} and the acceleration fields are the radiation or free fields because they vary like R^{-1}. The direction of propagation of the fields is defined to be the direction pointing from the position of the particle at the time of emission of the fields (the c-retarded position) to the position where the fields are observed at a later time. There also is no confusion that the acceleration fields are transverse fields, not because they are divergenceless—the acceleration E-field is not divergenceless—but because they are perpendicular to the directions of propagation of the fields at all observation points.

Let us consider a point-charged particle with charge q, trajectory $\mathbf{s}(t')$, velocity $\mathbf{v}(t')=d\mathbf{s}(t')/dt'$, and acceleration $\mathbf{a}(t')=d\mathbf{v}(t')/dt'$. The point of observation will be denoted by $\mathbf{r}$, and the observation time by t. The charge and current densities are $\rho(\mathbf{r}',t')=q\delta(\mathbf{r}'-\mathbf{s}(t'))$ and $\mathbf{J}(\mathbf{r}',t')=q\mathbf{v}(t')\delta(\mathbf{r}'-\mathbf{s}(t'))$. The potentials in the Lorenz gauge, also known as the Liénard-Wiechert potentials,[20] are

$$\Phi_c^{(L)}(\mathbf{r},t)=\int\frac{q\delta(\theta(\mathbf{r},t,t'))}{R(t')}dt' =\left\{\frac{q}{[1-c^{-1}\mathbf{u}(t')\cdot\mathbf{v}(t')]R(t')}\right\}_c, \tag{4.1a}$$

$$\mathbf{A}_c^{(L)}(\mathbf{r},t)=\int\frac{q\mathbf{v}(t')\delta(\theta(\mathbf{r},t,t'))}{cR(t')}dt' =\left\{\frac{q\mathbf{v}(t')}{c[1-c^{-1}\mathbf{u}(t')\cdot\mathbf{v}(t')]R(t')}\right\}_c, \tag{4.1b}$$

where $\theta(\mathbf{r},t,t')=t-R(t')/c-t'$, $\mathbf{R}(t')=\mathbf{r}-\mathbf{s}(t')$, $R(t')=|\mathbf{R}(t')|$, $\mathbf{u}(t')=\mathbf{R}(t')/R(t')=\nabla R(t')$, and $\{\ \}_c$ means that the quantities inside the braces are evaluated at the c-retarded time $t'=t-R(t')/c$.

The fields group naturally into the velocity and the acceleration fields:[20–22]

$$\mathbf{E}_c(\mathbf{r},t)=-\nabla\Phi_c^{(L)}(\mathbf{r},t)-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_c^{(L)}(\mathbf{r},t) =\mathbf{E}_c^{(\mathrm{vel})}(\mathbf{r},t)+\mathbf{E}_c^{(\mathrm{acc})}(\mathbf{r},t), \tag{4.2a}$$

$$\mathbf{B}_c(\mathbf{r},t)=\nabla\times\mathbf{A}_c^{(L)}(\mathbf{r},t)=\mathbf{B}_c^{(\mathrm{vel})}(\mathbf{r},t)+\mathbf{B}_c^{(\mathrm{acc})}(\mathbf{r},t), \tag{4.2b}$$

$$\mathbf{E}_c^{(\mathrm{vel})}(\mathbf{r},t)=\left\{\frac{q(1-v^2/c^2)(\mathbf{u}-\mathbf{v}/c)}{(1-\mathbf{u}\cdot\mathbf{v}/c)^3R^2}\right\}_c,$$
$$\mathbf{B}_c^{(\mathrm{vel})}(\mathbf{r},t)=\{\mathbf{u}\}_c\times\mathbf{E}_c^{(\mathrm{vel})}(\mathbf{r},t), \tag{4.3}$$

$$\mathbf{E}_c^{(\mathrm{acc})}(\mathbf{r},t)=\left\{\frac{q\mathbf{u}\times[(\mathbf{u}-\mathbf{v}/c)\times\mathbf{a}]}{c^2(1-\mathbf{u}\cdot\mathbf{v}/c)^3R}\right\}_c,$$
$$\mathbf{B}_c^{(\mathrm{acc})}(\mathbf{r},t)=\{\mathbf{u}\}_c\times\mathbf{E}_c^{(\mathrm{acc})}(\mathbf{r},t). \tag{4.4}$$

The significance of $\{\mathbf{u}\}_c$ is that it is the direction of propagation of the fields emitted at the c-retarded position $\mathbf{s}(t')$ at the c-retarded time t' when the observation is made at location $\mathbf{r}$ at observer's time t. This point is demonstrated in Eq. (4.4). The acceleration fields are transverse because they are perpendicular to the direction of propagation $\{\mathbf{u}\}_c$, that is, $\{\mathbf{u}\}_c\cdot\mathbf{E}_c^{(\mathrm{acc})}=\{\mathbf{u}\}_c\cdot\mathbf{B}_c^{(\mathrm{acc})}=0$. We have not seen any statements proclaiming the transversality of the acceleration fields based on their divergences. Actually the divergence of the acceleration E-field does not vanish:

$$\nabla\cdot\mathbf{E}_c^{(\mathrm{acc})}(\mathbf{r},t)=2q\left\{\frac{(1-v^2/c^2)(\mathbf{u}\cdot\mathbf{a})-(\mathbf{a}\cdot\mathbf{v}/c)(1-\mathbf{u}\cdot\mathbf{v}/c)}{c^2(1-\mathbf{u}\cdot\mathbf{v}/c)^4R^2}\right\}_c. \tag{4.5}$$

Also, $\{\mathbf{u}\}_c$ is the local direction of flow of radiation energy:

$$\mathbf{S}_c^{(\mathrm{acc})}(\mathbf{r},t)=\frac{c}{4\pi}\mathbf{E}_c^{(\mathrm{acc})}(\mathbf{r},t)\times\mathbf{B}_c^{(\mathrm{acc})}(\mathbf{r},t) =\frac{c}{4\pi}[\mathbf{E}_c^{(\mathrm{acc})}(\mathbf{r},t)]^2\{\mathbf{u}\}_c. \tag{4.6}$$

Because $\{\mathbf{u}\}_c$ is the direction of propagation of the fields, we say that a field or vector potential is longitudinal if it is parallel to $\{\mathbf{u}\}_c$ and transverse if it is perpendicular to $\{\mathbf{u}\}_c$.

Now we want to find a set of c-retarded potentials that generates only the acceleration fields and another that generates only the velocity fields.[8] For this purpose, we define two sets of potentials and their generating functions by using $(j)=(\mathrm{vel})$ or (acc),

$$\Gamma^{(\mathrm{vel})}(\mathbf{r},t')=\frac{q(1-v^2/c^2)}{(1-\mathbf{u}\cdot\mathbf{v}/c)R}, \tag{4.7}$$

$$\Gamma^{(\mathrm{acc})}(\mathbf{r},t')=\frac{q(\mathbf{u}\cdot\mathbf{a})}{c^2(1-\mathbf{u}\cdot\mathbf{v}/c)}, \tag{4.8}$$

$$\Phi_c^{(j)}(\mathbf{r},t) = \left\{ \frac{\Gamma^{(j)}(\mathbf{r},t')}{(1-\mathbf{u}\cdot\mathbf{v}/c)} \right\}_c,$$

$$\mathbf{A}_c^{(j)}(\mathbf{r},t) = \left\{ \frac{\mathbf{u}\Gamma^{(j)}(\mathbf{r},t')}{(1-\mathbf{u}\cdot\mathbf{v}/c)} \right\}_c. \tag{4.9}$$

We wish to show that the potentials $\mathbf{A}_c^{(\mathrm{vel})}(\mathbf{r},t)$ and $\Phi_c^{(\mathrm{vel})} \times(\mathbf{r},t)$ generate only the velocity fields, the potentials $\mathbf{A}_c^{(\mathrm{acc})}(\mathbf{r},t)$ and $\Phi_c^{(\mathrm{acc})}(\mathbf{r},t)$ generate only the acceleration fields, and these four potentials are related to the potentials in the Lorenz gauge by

$$\mathbf{A}_c^{(L)}(\mathbf{r},t) = \mathbf{A}_c^{(\mathrm{vel})}(\mathbf{r},t) + \mathbf{A}_c^{(\mathrm{acc})}(\mathbf{r},t) + \nabla \Lambda_c^{(\mathrm{vel\text{-}acc})}(\mathbf{r},t), \tag{4.10a}$$

$$\Phi_c^{(L)}(\mathbf{r},t) = \Phi_c^{(\mathrm{vel})}(\mathbf{r},t) + \Phi_c^{(\mathrm{acc})}(\mathbf{r},t) - \frac{1}{c}\frac{\partial}{\partial t}\Lambda_c^{(\mathrm{vel\text{-}acc})}(\mathbf{r},t), \tag{4.10b}$$

$$\Lambda_c^{(\mathrm{vel\text{-}acc})}(\mathbf{r},t) = -q\{\ln[(1-\mathbf{u}\cdot\mathbf{v}/c)R(t')]\}_c. \tag{4.10c}$$

To prove these statements, we first note that

$$-\nabla\Phi_c^{(j)} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}_c^{(j)} = \left\{ \frac{(-\nabla\Gamma^{(j)})}{(1-\mathbf{u}\cdot\mathbf{v}/c)} \right\}_c, \tag{4.11a}$$

$$\nabla\times\mathbf{A}_c^{(j)} = \left\{ \frac{\mathbf{u}\times(-\nabla\Gamma^{(j)})}{(1-\mathbf{u}\cdot\mathbf{v}/c)} \right\}_c = \{\mathbf{u}\}_c \times \left\{ \frac{(-\nabla\Gamma^{(j)})}{(1-\mathbf{u}\cdot\mathbf{v}/c)} \right\}_c. \tag{4.11b}$$

If we use Eq. (4.7) and $\nabla[(1-\mathbf{u}\cdot\mathbf{v}/c)R] = \nabla(R-\mathbf{R}\cdot\mathbf{v}/c) = (\mathbf{u}-\mathbf{v}/c)$, then

$$-\nabla\Gamma^{(\mathrm{vel})}(\mathbf{r},t') = \frac{q(1-v^2/c^2)(\mathbf{u}-\mathbf{v}/c)}{(1-\mathbf{u}\cdot\mathbf{v}/c)^2R^2}. \tag{4.12}$$

If we use Eq. (4.8) and $(1-\mathbf{u}\cdot\mathbf{v}/c) = \mathbf{u}\cdot(\mathbf{u}-\mathbf{v}/c)$, we obtain

$$\begin{aligned} -c^2\nabla\Gamma^{(\mathrm{acc})} &= -\nabla\left[\frac{q(\mathbf{R}\cdot\mathbf{a})}{(1-\mathbf{u}\cdot\mathbf{v}/c)R}\right] = -\frac{q\mathbf{a}}{(1-\mathbf{u}\cdot\mathbf{v}/c)R} \\ &\quad + \frac{q(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}-\mathbf{v}/c)}{(1-\mathbf{u}\cdot\mathbf{v}/c)^2R} \\ &= \frac{q(\mathbf{u}\cdot\mathbf{a})(\mathbf{u}-\mathbf{v}/c) - q[\mathbf{u}\cdot(\mathbf{u}-\mathbf{v}/c)]\mathbf{a}}{(1-\mathbf{u}\cdot\mathbf{v}/c)^2R} \\ &= \frac{q\mathbf{u}\times[(\mathbf{u}-\mathbf{v}/c)\times\mathbf{a}]}{(1-\mathbf{u}\cdot\mathbf{v}/c)^2R}. \end{aligned} \tag{4.13}$$

Thus, we have shown that the potentials $\mathbf{A}_c^{(\mathrm{vel})}$ and $\Phi_c^{(\mathrm{vel})}$ generate only the velocity fields, and $\mathbf{A}_c^{(\mathrm{acc})}$ and $\Phi_c^{(\mathrm{acc})}$ generate only the acceleration fields. The results in Eq. (4.10) can be shown easily; the proof will be left as an exercise for the reader.

V. CONCLUSIONS

The traditional approach to gauge transformations is based entirely on the properties of the gradient and curl components from the Helmholtz decomposition of the vector potentials. This purely static approach is not equipped to describe the most important dynamical property of the potentials: how and how fast the potentials propagate in space from the physical charge and current densities. Our approach corrects this deficiency.

As a consequence of our emphasis on the propagation behavior, we have learned that in the velocity gauge with $v \neq c$, the scalar potential propagates at speed v, and the vector potential has one v-propagating and one c-retarded component. The v-components of the scalar and vector potentials are unnecessary for the generation of the c-retarded fields. Because these v-components are detached from the c-retarded fields, they should not be involved in the definition of quantities intended to represent measurable observables unless such quantities are shown to be gauge-invariant first. Even when they are involved in gauge-invariant quantities, they have no effect on the measurement results because their only purpose is to cancel the other v-propagating effects of the gauge-invariant quantities.

Our results in Eqs. (3.14)–(3.17) with $v=\infty$ indicate that in the Coulomb gauge the vector potential has gradient components: the c-retarded gradient component is embedded in $\mathbf{A}_c^{(v)}$ in Eq. (3.16) and the instantaneous gradient component is $\mathbf{A}_v^{(v)}$ in Eq. (3.14).[1,4–7,13] Their divergences—not the physically independent gradient components themselves—cancel each other, causing the divergence of the vector potential to vanish.

Most importantly, we can now visualize the propagation of the potentials in the same way that we have visualized the propagation of the fields. Because we can visualize the potentials, we can relate the abstract concept of a gauge to the more concrete picture in which a gauge is represented by different components of its potentials propagating at different speeds in space. Once we have understood this relation clearly, we can relate the abstract concept of a violation of gauge invariance to the more concrete concept of a violation of causality and relativity.

A violation of gauge invariance means that there are bits and pieces of the potentials that do not cancel, resulting in the manifestation of the effects of these pieces of potentials. [An example of the potentials canceling cleanly is shown in Eqs. (3.12a) and (3.13a) with $v \neq c$. An example of the potentials not canceling cleanly is the classical canonical momentum in the v-gauge, $\mathbf{p}^{(v)} = m(d\mathbf{r}/dt) + (q/c)\mathbf{A}^{(v)}$, for a nonrelativistic particle with mass m, charge q, and velocity $d\mathbf{r}/dt$.] If we choose a gauge with potentials that carry faster-than-c components, we have a violation of causality and relativity because the effects of the potentials will exhibit themselves before the arrival of the c-retarded fields. Yang and Kobe[5] have used the gauge dependence of the conventional interaction Hamiltonian to show that the conventional interpretation of the quantum mechanical probabilities violates causality in those gauges with advanced potentials or faster-than-c retarded potentials. The link between violation of gauge invariance and violation of causality and relativity is the fundamental conceptual impetus behind the pursuit of a manifestly gauge-invariant interpretation of quantum mechanics in Refs. 14, 23, 24, 19, 25, 26, 5, and 27.

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constant source of inspiration in my pursuit of gauge invariance in classical and quantum mechanics. While a graduate student, the author discovered the velocity gauge—without any knowledge of Ref. 13—by studying Ref. 11. This gauge was a theme in a term paper submitted to Professor Byron (Ref. 28) for an atomic physics course in 1971. The other theme was on the violation of causality and relativity by the conventional quantum mechanical probabilities, which eventually became Ref. 5. The author thanks Professor Byron for the constant encouragement given to him during his graduate student years.

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[10] Of all the textbooks on classical electromagnetism, Jackson's text has the most systematic and in-depth discussions on gauge transformations. In addition to the third edition listed in Ref. 3, we also list the previous two editions because we can see the trend in our understanding on this topic and because they were most relevant when the author discovered the propagation property of the potentials.

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[16] Sometimes, a poorly formulated constraint may not lead to a possible solution from Eq. (2.6). For example, the constraint $\nabla\cdot\mathbf{A}(\mathbf{r},t)=0$ and $\Phi(\mathbf{r},t)=0$ for all $\mathbf{r}$ and t cannot be satisfied by an arbitrary charge and current distribution. The reader can easily show that the condition $\Phi(\mathbf{r},t)=0$ contradicts the condition $\nabla\cdot\mathbf{A}(\mathbf{r},t)=0$ except if $\rho(\mathbf{r},t)$ vanishes identically.

[17] We briefly mention the traditional use of gauge transformations of the first and second kinds. We use $\Psi(\mathbf{r},t)$ to denote the wave function of the Schrödinger equation with potentials $\mathbf{A}$ and Φ and $\Psi'(\mathbf{r},t)$ for the wave function with potentials $\mathbf{A}'$ and Φ', where $(\mathbf{A}',\Phi')$ and $(\mathbf{A},\Phi)$ are related by the gauge transformation in Eq. (2.7) by a real gauge function $\chi(\mathbf{r},t)$. Then, the quantum mechanical relation, $\Psi'=\exp[(iq/c\hbar)\chi]\Psi$, is called a gauge transformation of the first kind. The electromagnetic gauge transformation in Eq. (2.7) is called a gauge transformation of the second kind. See, for example, J. J. Sakurai, *Advanced Quantum Mechanics* (Addison-Wesley, Reading, 1967), pp. 11–15.

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[20] In Refs. 3, 11, and 12, Chap. 14.

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