ON THE SCATTERING OF SPHERICAL WAVES BY A CYLINDRICAL OBJECT

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Summary

The connection between plane wave and spherical wave scattering from an infinitely cylindrical object is investigated. If the distance of the source and the observer from the axis of the cylinder are denoted by ϱ_0 and ϱ, respectively, the ratio of the scattered field to that for plane wave excitation $[\varrho_0/(\varrho_0 + \varrho)]^{\frac{1}{2}}$.

In the theoretical treatment of problems in electromagnetic and acoustic scattering it is usual to assume that the primary excitation can be represented by a uniform plane wave. This is well justified if the largest dimension of the scattering obstacle is small compared to the distance to the source. In the case of scattering from long cylindrical obstacles, however, the curvature of the wave front of the incident wave cannot always be neglected. It is then of interest to examine if there is any simple relation between plane wave and spherical wave scattering from cylindrical obstacles.

With respect to a cylindrical coordinate system (ϱ, φ, z), the cylindrical obstacle is taken to be infinite in the z direction with uniform cross-section bounded by the surface $\varrho = a$. It is assumed to consist of isotropic and linear electrical or acoustical properties. The source at $(\varrho_0, \varphi_0, 0)$ is considered to be an axially directed electric dipole in the electromagnetic case or a point source in the acoustic case. The primary excitation is then represented by the function ψ^P, which is the z component of the electromagnetic Hertz vector or the acoustic velocity potential, and can be written

$$\psi^P = C e^{-i\beta r}/r \tag{1}$$

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where $\beta = 2\pi/\text{wavelength}$, and $r = (\bar{\varrho}^2 + z^2)^{\frac{1}{2}}$ with

$$\bar{\varrho} = [\varrho^2 + \varrho_0^2 - 2\varrho\varrho_0 \cos(\varphi - \varphi_0)]^{\frac{1}{2}}$$

and C is a constant. The time factor implied is exp $(i\omega t)$. Now making use of Sommerfeld's integral [1]) it is seen that

$$\psi^P = -\tfrac{1}{2}Ci \int_{-\infty}^{+\infty} H_0^{(2)}[(\beta^2 - g^2)^{\frac{1}{2}}\bar{\varrho}]\mathrm{e}^{-igz}\,\mathrm{d}g, \tag{2}$$

and by employing the addition theorem [2]) for the Hankel function $H_0^{(2)}$ it follows that

$$\psi^P = -\tfrac{1}{2}Ci \sum_{m=-\infty}^{+\infty} \mathrm{e}^{im(\varphi-\varphi_0)} \int_{-\infty}^{+\infty} H_m^{(2)}(u\varrho_0) J_m(u\varrho)\mathrm{e}^{-igz}\,\mathrm{d}g. \tag{3}$$

for $\varrho < \varrho_0$ and $u = (\beta^2 - g^2)^{\frac{1}{2}}$. The excitation can now be interpreted as a spectrum of cylindrical waves of order m. Each of these cylindrical harmonics will be scattered by the cylindrical obstacle and in general will excite a spectrum of cylindrical waves of order n, of the form

$$R_{nm}\, \mathrm{e}^{in\varphi} H_n^{(2)}(u\varrho) \text{ for } n = 0, \pm 1, \pm 2, \ldots .$$

The factor R_{nm} is a cylindrical reflection coefficient which relates the scattered cylindrical waves of order n to a given incident cylindrical wave of order m. The scattered field ψ^s can then be written

$$\psi^s = -\tfrac{1}{2}Ci \sum_{m=-\infty}^{+\infty} \mathrm{e}^{im(\varphi-\varphi_0)} \int_{-\infty}^{+\infty} H_m^{(2)}(u\varrho_0) \sum_{n=-\infty}^{+\infty} R_{mn}\, \mathrm{e}^{i(n-m)\varphi} H_n^{(2)}(u\varrho)\mathrm{e}^{-igz}\,dg \tag{4}$$

The factor R_{mn} is to be determined by the properties and the form of the cylinder [3]). In the case of a perfectly conducting cylinder of radius a (circular cross-section) it can be seen that

$$R_{mm} = -\frac{J_m(ua)}{H_m(ua)} \text{ and } R_{nm} = 0,\ n \neq m, \tag{5}$$

and for the rigid circular cylinder (acoustic case)

$$R_{mm} = -\left[\frac{\partial J_m(x)/\partial x}{\partial H_m^{(2)}(x)/\partial x}\right]_{x=ua} \text{ and } R_{nm} = 0,\ n \neq m \tag{6}$$

In fact, for cylindrical obstacles with polar symmetry, R_{nm} for $n \neq m$, is always identically zero[4]). In the more general case R_{nm} is finite for $n \neq m$ and is a function of the axial propagation constant g.

The scattered fields, in the practical instance, are always observed at larger distances from the cylinder (i.e. $\varrho \gg a$) and furthermore the source is located at a large distance from the cylinder (i.e. $\varrho_0 \gg a$). Under these conditions the infinite integral in equation (4) is of the form

$$J = -\tfrac{1}{2}i \int_{-\infty}^{+\infty} T(g)\, H_m^{(2)}(u\varrho_0)\, H_n^{(2)}(u\varrho) e^{-igz}\, dg, \tag{7}$$

where $T(g)$ is a slowly varying compared with $H_m^{(2)}$ and $H_n^{(2)}$ as a function of g. In fact since $u\varrho_0$ and $u\varrho$ are large compared to one for the significant values of g, the Hankel functions can be represented by the first term of their asymptotic expansion. That is

$$H_m^{(2)}(u\varrho_0) \simeq (2/\pi u\varrho_0)^{\frac{1}{2}}\, e^{-iu\varrho_0}\, e^{i(\frac{1}{2}m\pi + \frac{1}{4}\pi)} \tag{8}$$

and similarly for $H_n^{(2)}(u\varrho)$.

It is now convenient to change the integration variable to a defined by

$$g = \beta \cos a$$

and therefore

$$u = \beta \sin a.$$

The integral can now be expressed as

$$J = -\frac{e^{\frac{1}{2}i(m+n)\pi}}{\pi(\varrho\varrho_0)^{\frac{1}{2}}} \int_C T(\beta \cos a)\, e^{-i\beta R\cos(\theta - a)}\, da, \tag{9}$$

where

$$R = [(\varrho + \varrho_0)^2 + z^2]^{\frac{1}{2}} \text{ and } \theta = \tan^{-1}\left(\frac{\varrho + \varrho_0}{z}\right).$$

The contour C in the a plane is from $(\pi, -i\infty)$ along a line parallel to the negative imaginary axis to $(\pi, 0)$, then along the real axis to the origin $(0, 0)$ and finally out along the positive imaginary axis to $+i\infty$. The integral is now in a form to be evaluated by the method of steepest descents. This process can be carried out conveniently by introducing a new variable of integration χ defined by

$$\cos(\theta - a) = 1 - i\chi^2$$

where χ is taken to range from $-\infty$ to $+\infty$ through real values. The consequent deformation of the path of integration does not change the value of the original integral unless some poles of the integrand are swept over. The contribution from these poles are associated wtih guided or surface waves. The fields associated with them are attenuated very rapidly with distance from the axis of the cylinder. It is therefore reasonable to assume that the contribution to the far zone scattered fields from any swept over poles can be neglected since ϱ_0 and ϱ are already assumed large compared to the wavelength. The integral can now be expressed by

$$J \cong e^{-\frac{1}{4}i\pi} e^{\frac{1}{2}i(m+n)\pi} (2/\varrho\varrho_0)^{\frac{1}{2}}\pi^{-1} e^{-i\beta R} \int_{-\infty}^{+\infty} \frac{e^{-\beta R\chi^2} T(\beta \cos a)\, d\chi}{(1 - \frac{1}{2}i\chi^2)^{\frac{1}{2}}}. \quad (10)$$

If the factor $(1 - \frac{1}{2}i\chi^2)^{-\frac{1}{2}}T$ is expanded in a power series in χ^2 about $\chi = 0$, the integrations can be carried out. Retaining only the first term,

$$J \simeq e^{\frac{1}{2}i(m+n)\pi} e^{\frac{1}{4}i\pi} \left[\frac{2}{\pi\beta R\varrho\varrho_0}\right]^{\frac{1}{2}} e^{-i\beta R}\, T(\beta \cos \theta). \quad (11)$$

Employing this result, it finally follows that

$$\psi^s \simeq C e^{\frac{1}{4}i\pi} e^{-i\beta R} \frac{2}{(\pi\beta R\varrho\varrho_0)^{\frac{1}{2}}} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} e^{\frac{1}{2}i(m+n)\pi} e^{i(n\varphi - m\varphi_0)} R_{mn}(\beta \cos\theta), \quad (12)$$

$$\psi^s = \psi_0\, e^{-i\beta(z-z_0)\cos\theta} \left[\frac{2i}{\pi\beta\varrho \sin\theta}\right]^{\frac{1}{2}} \left[\frac{\varrho_0}{\varrho + \varrho_0}\right]^{\frac{1}{2}}$$

$$\sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} e^{\frac{1}{2}i(m+n)\pi} e^{i(n\varphi - m\varphi_0)} R_{mn}(\beta \cos\theta)\, e^{-i\beta\varrho\sin\theta}, \quad (13)$$

where

$$\psi_0 = \frac{C e^{-i\beta(\varrho_0^2 + z_0^2)^{\frac{1}{2}}}}{(\varrho_0^2 + z_0^2)^{\frac{1}{2}}} \quad \text{and} \quad z_0 = \frac{\varrho_0}{t\varrho\theta}$$

It can be recognized that ψ_0 is the value of the incident field at $(z = z_0)$ on the axis of the cylinder. This is the point of specular reflection from the viewpoint of geometrical optics. The form of the scattered fields for the case of plane wave incidence can be obtained directly. For example, the primary excitation can be expressed as

$$\psi^P = \psi_0\, e^{i\beta\varrho\cos(\varphi-\varphi_0)\sin\theta}\, e^{-i\beta z\varrho\cos\theta}, \quad (14)$$

where ψ_0 is the amplitude. Employing an addition theorem, this can be written

$$\psi^P = \psi_0\, e^{-i\beta z\cos\theta} \sum_{m=-\infty}^{+\infty} e^{\frac{1}{2}im\pi} J_m(\beta\varrho \sin\theta)\, e^{im(\varphi-\varphi_0)}. \quad (15)$$

The scattered field is then given by

$$\psi^s = \psi_0 \, e^{-i\beta z \cos\theta} \sum_{m=-\infty}^{+\infty} e^{im(\varphi-\varphi_0)} \, e^{\frac{1}{2}im\pi} \sum_{n=-\infty}^{+\infty} R_{mn}(\beta \cos\theta) e^{i(n-m)\varphi} H_n^{(2)}(\beta\varrho \sin\theta), \quad (16)$$

which in the far field, assumes the form

$$\psi^s = \psi_0 \, e^{-i\beta z \cos\theta} \left[\frac{2i}{\pi\beta\varrho \sin\theta}\right]^{\frac{1}{2}} . \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} e^{\frac{1}{2}i(m+n)\pi} \, e^{i(n\varphi - m\varphi_0)} R_{mn}(\beta \cos\theta) \, e^{-i\beta\varrho \sin\theta}. \quad (17)$$

It is now immediately apparent that the structure of the scattered fields for spherical wave and plane wave scattering are very similar. In fact on comparing equations (13) and (17), it is seen that the spherical wave scattering amplitude is $[\varrho_0/(\varrho + \varrho_0)]^{\frac{1}{2}}$ times the plane wave scattering wave amplitude. It is noted that if $\varrho_0 \gg \varrho$, the factor approaches unity.

Received 11th June, 1955.

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