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Field of a Particle in Uniform Motion and Uniform Acceleration[*]

NATHAN ROSEN

Department of Physics, Israel Institute of Technology, Haifa, Israel

The line element in the neighborhood of a charged particle, according to general relativity theory, is put into the isotropic form. By means of a Lorentz transformation the gravitational and electromagnetic fields of a particle in uniform motion are obtained. By transforming to a uniformly accelerated system one obtains the fields of a particle accelerated in a uniform gravitational field. From the form of the solution it can be concluded that a charged particle in a uniform gravitational field does not radiate.

I. ISOTROPIC SOLUTION

The aim of this paper is to present some solutions of the Einstein and of the Einstein–Maxwell field equations describing the gravitational and electromagnetic field of a single particle in certain simple states of motion.

The Nordström solution of the field equations is well known. For the case of a particle of mass m and charge e, at rest at the origin and possessing spherical symmetry, the line element of general relativity theory can be written in the form[1]

$$ds^2 = -\sigma^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + \sigma dt^2, \tag{1}$$

where

$$\sigma = 1 - (2m/r) + (e^2/r^2), \tag{2}$$

and $r, \theta, \phi = x^1, x^2, x^3$ are spherical polar coordinates and $t = x^4$ is the time. The electromagnetic potential is given by

$$\phi_4 = e/r, \qquad \phi_1 = \phi_2 = \phi_3 = 0. \tag{3}$$

It will be convenient to put this solution into the isotropic form. For this

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[1] The usual units of general relativity are used, with $c, k = 1$.

purpose we set

$$r = R\left[\left(1+\frac{m}{2R}\right)^2 - \frac{e^2}{4R^2}\right], \qquad (4)$$
$$= R + m - (\alpha^2/R),$$

where

$$\alpha^2 = \tfrac{1}{4}(e^2 - m^2). \qquad (5)$$

It then follows readily that

$$ds^2 = -\lambda(R)[dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)] + \mu(R)dt^2, \qquad (6)$$

where

$$\lambda(R) = \left(1 + \frac{m}{R} - \frac{\alpha^2}{R^2}\right)^2, \qquad (7)$$

and

$$\mu(R) = \sigma(r) = \frac{[1 + (\alpha^2/R^2)]^2}{[1 + (m/R) - (\alpha^2/R^2)]^2}. \qquad (8)$$

The electrostatic potential then takes the form

$$\phi_4 = \phi(R), \qquad (9)$$

where

$$\phi(R) = \frac{e}{R[1 + (m/R) - (\alpha^2/R^2)]}. \qquad (10)$$

In the case of a neutral particle ($e = 0$), (6) takes on the well-known isotropic form of the Schwarzschild solution,

$$ds^2 = -\left(1 + \frac{m}{2R}\right)^4 [dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\phi^2)] + \left[\frac{1 - (m/2R)}{1 + (m/2R)}\right]^2 dt^2. \qquad (11)$$

In the case of a particle for which $e = m$ ($\alpha = 0$), (6) takes on the form

$$ds^2 = -[1 + (m/R)]^2[dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)] + [1 + (m/R)]^{-2}dt^2. \qquad (12)$$

From the mathematical standpoint, one might also consider the case $m = 0$. We would then have

$$ds^2 = -\left(1 - \frac{e^2}{4R^2}\right)^2 [dR^2 + R^2(d\phi^2 + \sin^2\theta\, d\phi^2)] + \left[\frac{1 - (e^2/4R^2)}{1 - (e^2/4R^2)}\right]^2 dt^2. \qquad (13)$$

In the general case, introducing "Cartesian coordinates", X, Y, Z, by the usual relations, and writing $T = t$, we get

$$ds^2 = -\lambda(R)(dX^2 + dY^2 + dZ^2) + \mu(R)dT^2, \qquad (14)$$

where

$$R^2 = X^2 + Y^2 + Z^2. \qquad (15)$$

Taking the electrostatic field as described by the potential given by Eqs. (9) and (10), one finds that for the electromagnetic field tensor

$$F_{\mu\nu} = \phi_{\mu,\nu} - \phi_{\nu,\mu}, \qquad (16)$$

the nonvanishing components are given by

$$F_{4k} = -F_{k4} = \phi' X^k/R \qquad (k = 1, 2, 3), \qquad (17)$$

where $X^1, X^2, X^3, X^4 = X, Y, Z, T$, and $\phi' = d\phi/dR$. For the energy-momentum density tensor,

$$T_{\mu\nu} = (1/4\pi)[\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F_{\mu\alpha}F_\nu{}^\alpha], \qquad (18)$$

the nonvanishing components are found to be

$$\begin{aligned} T_{kl} &= (1/8\pi\mu)\phi'^2[\delta_{kl} - 2X^kX^l/R^2] \quad (k, l = 1, 2, 3), \\ T_{44} &= (1/8\pi\lambda)\phi'^2. \end{aligned} \qquad (19)$$

II. PARTICLE IN UNIFORM MOTION

We take the basic solution in the form given by Eqs. (7)–(10), (14), and (15). In order to describe the field of a particle which is moving with a constant velocity, we go over by means of a Lorentz transformation to a reference system moving with an equal but opposite velocity. Denoting the coordinate in this system by x, y, z, t, we take as the Lorentz transformation

$$\begin{aligned} X &= \gamma(x - vt), \\ Y &= y, Z = z, \\ T &= \gamma(t - vx), \end{aligned} \qquad (20)$$

where

$$\gamma = (1 - v^2)^{-1/2}. \qquad (21)$$

This corresponds to the case in which the particle in the new coordinate system has a velocity v in the direction of the x-axis. Substituting into (14), one gets

$$ds^2 = -\frac{\lambda - v^2\mu}{1 - v^2}dx^2 - \lambda(dy^2 + dz^2) + \frac{\mu - v^2\lambda}{1 - v^2}dt^2 + \frac{2(\lambda - \mu)v}{1 - v^2}dx\,dt. \qquad (22)$$

Here, $\lambda = \lambda(R)$, $\mu = \mu(R)$ (as given by Eqs. (7) and (8)), and

$$R^2 = \gamma^2(x - vt)^2 + y^2 + z^2. \tag{23}$$

For sufficiently large values of R one can expand λ and μ in powers of $1/R$. Keeping second-order terms, one then gets (with $x^1, x^2, x^3, x^4 = x, y, z, t$) the following nonvanishing components for the metric tensor:

$$\begin{aligned}
g_{11} &= -1 - \frac{2m}{R}\left(\frac{1+v^2}{1-v^2}\right) + \frac{1}{2R^2(1-v^2)}\,[e^2(1+2v^2) + m^2(-3+4v^2)],\\
g_{14} &= \frac{v}{1-v^2}\left[\frac{4m}{R} - \frac{3e^2+m^2}{2R^2}\right],\\
g_{22} &= g_{33} = -1 - \frac{2m}{R} + \frac{e^2-3m^2}{2R^2},\\
g_{44} &= 1 - \frac{2m}{R}\left(\frac{1+v^2}{1-v^2}\right) + \frac{1}{2R^2(1-v^2)}\,[e^2(2+v^2) + m^2(4-3v^2)].
\end{aligned} \tag{24}$$

The electromagnetic potential is now given by

$$\phi_1 = -\phi(R)\gamma v, \qquad \phi_2 = \phi_3 = 0, \qquad \phi_4 = \phi(R)\gamma, \tag{25}$$

where $\phi(R)$ is defined by Eq. (10). The tensors $F_{\mu\nu}$ and $T_{\mu\nu}$ can be obtained either from (25) or by applying the Lorentz transformation to the components given by (17) and (19).

III. PARTICLE IN UNIFORM ACCELERATION

We now consider the case in which the particle is being accelerated in a uniform gravitational field. By the equivalence principle, this situation is indistinguishable from that in which the reference frame is uniformly accelerated. For the latter case we take as the coordinate transformations (*1*)

$$\begin{aligned}
X &= [x - (1/g)]\cosh gt + (1/g),\\
Y &= y, \qquad Z = z,\\
T &= [-x + (1/g)]\sinh gt,
\end{aligned} \tag{26}$$

where $-g$ is the (constant) proper acceleration, and is taken to be along the X-axis. In the new coordinate system there is a gravitational field g along the x-axis. It should be noted that for $X = 0$, which corresponds to the position of the particle, x decreases from $1/g$ (for $t = -\infty$) to zero (for $t = 0$) and then increases again to $1/g$ (for $t = +\infty$).

Substituting (26) into (14), we obtain an expression for the line-element corresponding to the following nonvanishing components of the metric tensor

(with $x^1, x^2, x^3, x^4 = x, y, z, t$):

$$\begin{aligned}
g_{11} &= \mu\sinh^2 gt - \lambda\cosh^2 gt,\\
g_{14} &= (\lambda - \mu)(1 - gx)\sinh gt\cosh gt,\\
g_{22} &= g_{33} = -\lambda,\\
g_{44} &= (\mu\cosh^2 gt - \lambda\sinh^2 gt)(1 - gx)^2.
\end{aligned} \tag{27}$$

Here again $\lambda = \lambda(R)$, $\mu = \mu(R)$, where now however

$$R^2 = \{[x - (1/g)]\cosh gt + (1/g)\}^2 + y^2 + z^2. \tag{28}$$

For sufficiently large values of R one can expand the metric tensor in powers of $1/R$, and one then gets, keeping quantities of the second order, the following expressions for the components:

$$\begin{aligned}
g_{11} &= -1 - (2m/R)(1 + 2\sinh^2 gt)\\
&\qquad + (1/2R^2)[e^2(1 + 3\sinh^2 gt) + m^2(-3 + \sinh^2 gt)],\\
g_{14} &= [(4m/R) - (1/2R^2)(3e^2 + m^2)](1 - gx)\sinh gt\cosh gt,\\
g_{22} &= g_{33} = -1 - (2m/R) + (1/2R^2)(e^2 - 3m^2),\\
g_{44} &= \{1 - (2m/R)(1 + 2\sinh^2 gt)\\
&\qquad + (1/2R^2)[e^2(2 + 3\sinh^2 gt) + m^2(4 + \sinh^2 gt)]\}(1 - gx)^2.
\end{aligned} \tag{29}$$

For sufficiently small values of g, one can of course replace $\sinh gt$ by gt and $\cosh gt$ by unity.

The electromagnetic potential can be readily calculated and is found to have the components

$$\begin{aligned}
\phi_1 &= -\phi(R)\sinh gt,\\
\phi_4 &= \phi(R)(1 - gx)\cosh gt,
\end{aligned} \tag{30}$$

where $\phi(R)$ is given by (10). The electromagnetic field intensity is given by

$$\begin{aligned}
F_{41} &= \phi'(X/R)(1 - gx),\\
F_{42} &= \phi'(Y/R)(1 - gx)\cosh gt,\\
F_{43} &= \phi'(Z/R)(1 - gx)\cosh gt,\\
F_{21} &= \phi'(Y/R)\sinh gt,\\
F_{31} &= \phi'(Z/R)\sinh gt.
\end{aligned} \tag{31}$$

The energy-momentum density tensor is found to have the following components

$$\begin{aligned}
T_{11} &= \frac{\phi'^2}{8\pi}\left[\frac{1}{\lambda}\sinh^2 gt + \frac{1}{\mu}\left(1 - \frac{2X^2}{R^2}\right)\cosh^2 gt\right],\\
T_{12} &= -\frac{\phi'^2}{4\pi\mu}\frac{XY}{R^2}\cosh gt,\\
T_{13} &= -\frac{\phi'^2}{4\pi\mu}\frac{XZ}{R^2}\cosh gt,\\
T_{22} &= \frac{\phi'^2}{8\pi\mu}\left(1 - \frac{2Y^2}{R^2}\right),\\
T_{23} &= -\frac{\phi'^2}{4\pi\mu}\frac{YZ}{R^2},\\
T_{33} &= \frac{\phi'^2}{8\pi\mu}\left(1 - \frac{2Z^2}{R^2}\right),\\
T_{14} &= -\frac{\phi'^2}{8\pi}\left[\frac{1}{\lambda} + \frac{1}{\mu}\left(1 - \frac{2X^2}{R^2}\right)\right](1 - gx)\sinh gt\cosh gt,\\
T_{24} &= \frac{\phi'^2}{4\pi\mu}\frac{XY}{R^2}(1 - gx)\sinh gt,\\
T_{34} &= \frac{\phi'^2}{4\pi\mu}\frac{XZ}{R^2}(1 - gx)\sinh gt\\
T_{44} &= \frac{\phi'^2}{8\pi}\left[\frac{1}{\lambda}\cosh^2 gt + \frac{1}{\mu}\left(1 - \frac{2X^2}{R^2}\right)\sinh^2 gt\right](1 - gx)^2.
\end{aligned}\tag{32}$$

In Eqs. (31) and (32) one makes use of the transformation equations (26) in order to obtain the components as functions of x, y, z, t.

IV. RADIATION

In the preceding section the gravitational and electromagnetic fields have been obtained for a particle in a homogeneous gravitational field which has an intensity g in the direction of the x-axis. The particle therefore experiences an acceleration g in this direction. The conditions considered are such that, for negative values of t, the particle is moving along the x-axis in a negative direction toward the origin. It reaches the origin at $t = 0$, and then reverses its motion, acquiring a positive velocity for $t > 0$.

It should be remarked that the solution for the gravitational and electromagnetic fields of the particle, obtained by the use of the equivalence principle, is uniquely determined. Thus, one does not have any ambiguity here corresponding to that arising when one solves the Maxwell equations for an accelerated particle using advanced and retarded potentials.

It should also be pointed out that the components of the energy-momentum density tensor $T_{\mu\nu}$, as given by Eq. (32), are characterized by the property that they are all even functions of t, except for those having one index equal to 4, which are odd. This is also true for the contravariant and mixed components, as one can readily verify. If one compares the situation at two times, t and $-t$, for which the particle is at the same position (and has equal but opposite velocities), one finds therefore that the distributions of stresses and energy density are the same, but the energy current density (Poynting vector) differs in sign at each point. One sees then that the energy distribution in the field is the same at the two times, while the kinetic and potential energies of the particle are also unchanged, leading one to conclude that *no radiation of energy has taken place* between these times.

One therefore arrives at the conclusion that a charged particle, uniformly accelerated by a gravitational field, does not radiate. This appears to be in contradiction to the results obtained by Fulton and Rohrlich (*2*).[2]

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REFERENCES

1. C. MØLLER, "The Theory of Relativity", p. 255. Oxford Univ. Press, Oxford, 1952.
2. T. FULTON AND F. ROHRLICH, *Ann. Phys.* **9**, 499 (1960).

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[2] Reference *2* contains references to other works on this question.