Physics. — *"Note on the circumstance that an electric charge moving in accordance with quantum-conditions does not radiate."* By G. NORDSTRÖM. Supplement N°. 43*a* to the Communications from the Physical Laboratory at Leiden. (Communicated by H. KAMERLINGH ONNES).

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One often hears the remark: BOHR's atomic theory is at variance with classical electrodynamics in assuming that an electron which is moving according to quantum-conditions does not radiate energy in the form of electromagnetic waves. The assertion formulated in this way does not seem to me to state correctly where the opposition between BOHR's assumption and classical electrodynamics lies. In the sequel I shall try to substantiate this view. We shall begin by looking at the problem from a general point of view.

If an otherwise empty space contains electric charges whose motions are completely fixed, the electro-magnetic field is not singly determined by means of the MAXWELL-LORENTZ field-equations. In order to obtain a perfectly definite condition certain boundary-conditions must be fixed and it is to these that we shall give our attention. Whatever the field may be, it may be represented by the electro-dynamic potentials viz. the vector-potential $\mathfrak{A}$ and the scalar potential φ, which may also be combined in a four-dimensional vector-potential with the following components:

$$\varphi_x = \mathfrak{A}_x \quad , \quad \varphi_y = \mathfrak{A}_y \quad , \quad \varphi_z = \mathfrak{A}_z \quad , \quad \varphi_t = \varphi \quad . \quad . \quad . \quad 1)$$

The potentials determine the field-vectors $\mathfrak{E}$, $\mathfrak{B}$ completely by means of the equations [1])

$$\left.\begin{aligned} \mathfrak{B} &= rot\, \mathfrak{A}, \\ \mathfrak{E} &= -grad\, \varphi - \frac{1}{c}\frac{\partial \mathfrak{A}}{\partial t} \quad (c = \text{velocity of light}) \end{aligned}\right\} \quad . \quad . \quad . \quad 2)$$

On the other hand the potentials $\mathfrak{A}$, φ are not completely determined by the field. For this reason we may submit them to the condition

$$div\, \mathfrak{A} + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0,$$

[1]) Comp. for instance M. ABRAHAM, Theorie der Elektrizität II, 2[te] Aufl., p. 36.

and we then obtain for the various potential-components (using rational units) the following differential equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi_\alpha}{\partial y^2} + \frac{\partial^2 \varphi_\alpha}{dz^2} - \frac{1}{c^2} \frac{\partial^2 \varphi_\alpha}{\partial t^2} = -\sigma_\alpha, \quad . \quad . \quad . \quad . \quad (3)$$

$$(\alpha = x, y, z, t).$$

In this equations σ_x, σ_y, σ_z are the components of the electric current, σ_t is the density of the electricity. The motion of the charges being given, σ_x, σ_y, σ_z, σ_t are known functions of x, y, z, t. The general solution of the differential equations (3) may then be put in the following form:

Let x_0, y_0, z_0 be the coordinates of a point for which φ_α is to be found. The distance of this point to a point (x, y, z) may be called r, so that

$$r^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The sign of r is not fixed by this relation, but we may leave it undetermined in the mean time.

A unit vector with components

$$\frac{x - x_0}{r}, \ \frac{y - y_0}{r}, \ \frac{z - z_0}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

will be represented by $\mathfrak{r}$. The direction of $\mathfrak{r}$ is evidently dependent upon the choice of the sign of r. Let F be an arbitrary closed surface which encloses the point x_0, y_0, z_0. A surface-element of F, considered as a vector directed outwards, will be denoted by $d\mathfrak{F}$. Using these symbols we may write the general integral of the differential equation (3) in the form [1]:

$$\varphi_\alpha(x_0, y_0, z_0, t) = \pm \frac{1}{4\pi} \left\{ \int dV \frac{\sigma_\alpha}{r} + \int d\mathfrak{F} \left(\frac{1}{r} \, grad \, \varphi_\alpha + \frac{\mathfrak{r}}{cr} \frac{\partial \varphi_\alpha}{\partial t} + \frac{\mathfrak{r}}{r^2} \varphi \right) \right\}_{t - \frac{r}{c}} . (6)$$

The surface-integral has to be extended over the closed surface F, the space-integral over the enclosed space V containing the point x_0, y_0, z_0. The index $t - \frac{r}{c}$ is meant to indicate, that at the right-hand side the quantities σ_α, φ_α, grad φ_α, $\frac{\partial \varphi_\alpha}{\partial t}$ refer to the time $t - \frac{r}{c}$, which varies from point to point. The double sign on the right-hand

[1]) A proof of equation (6) may be found in Finska Vetenskapssocietetens Förhandl. L. I. 1908—09, Afd. A. n⁰. 6. If the sign of r is not fixed beforehand, it is easily found that the right-hand side has the double sign.

side is required as long as the sign of r has not been settled, but may be chosen at will; the $+$ sign holds for positive, the $-$ sign for negative r. If the sign is taken positive, the moment $t - \frac{r}{c}$ is earlier than t and φ_α must then be taken as a delayed potential. If the negative sign is chosen, the moment $t - \frac{r}{c}$ is later than t and φ_α is to be considered as an advanced potential.

Every function $\varphi_\alpha(x, y, z, t)$, *therefore, which satisfies the differential equation* (3), *may be considered either as a delayed or as an advanced potential, if only the contribution to the potential which is due to the boundary surface F (which may also be moved to infinity) is taken into account.*

It follows that every electro-magnetic field, i. e. every field for which the MAXWELL-LORENTZ equations hold, may be calculated for a moment t either from the condition at the time $t - \frac{(r)}{c}$ or from that at the time $t + \frac{(r)}{c}$, if only the contribution by the boundary surface is taken into account. This contribution is necessarily different in the two cases.

If the surface F is made to move to infinity and if at the same time the condition is laid down, that at the boundary the surface-integral has the value zero, if the potential is considered as a delayed potential, the ordinary solution is obtained of the problem to calculate the field from the charges. But this solution is only one particular one: there are an infinite number of others.

The author may be excused for having discussed this question at some length: it seemed to him that it is not always sufficiently kept in view.

We shall now prove, that every periodical motion of electricity allows the assumption of a field such that no energy is radiated. The separate points of the electric charges will be identified by means of 3 parameters ξ, η, ζ. Every set of values ξ, η, ζ, therefore, denotes a definite point in the electricity sharing the motion of the latter. The motion is completely described by the equations

$$\left.\begin{aligned} x &= x(\xi, \eta, \zeta, t), \\ y &= y(\xi, \eta, \zeta, t), \\ z &= z(\xi, \eta, \zeta, t), \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (7)$$

that is to say, for given values of ξ, η, ζ the coordinates x, y, z are functions of the time. Let us consider the motion represented by equations (7) as being completely given. This motion we shall call *motion* 1. We therefore assume, that x, y, z are known functions of ξ, η, ζ, t for all values of t from $-\infty$ to $+\infty$. We then calculate the electro-magnetic field by means of *delayed* potentials and choose the boundary conditions in such a manner, that the surface-integral in (6) becomes zero for each potential-component φ_α, when the surface F moves to infinity. The field is then singly determined by the motion of the electricity. We shall call the field obtained *field* 1. In this case we obviously have a radiation of energy.

We shall further consider a different motion of the electricity, *motion* 2, which is obtained from motion 1 by reversing the sign of t.

$$\left.\begin{aligned} x &= x(\xi, \eta, \zeta, -t), \\ y &= y(\xi, \eta, \zeta, -t), \\ z &= z(\xi, \eta, \zeta, -t), \end{aligned}\right\} \quad \text{motion 2.}$$

In this system all paths are evidently described in a direction opposite to that of motion 1. For motion 2 we again calculate the electro-magnetic field by means of *delayed* potentials and with the same boundary-conditions as before. We shall again obtain a field with energy-radiation, which we shall call *field* 2.

If the motion 1 is periodical, this will also be the case for motion 2 and the radiation during one period is equal for field 1 and field 2. We now pass from field 2 to a new field 3, by reversing the sign of t and at the same time the components of the magnetic field $\mathfrak{B}_x, \mathfrak{B}_y, \mathfrak{B}_z$. It is easily shown, that with this change of sign the MAXWELL-LORENTZ field-equations remain valid. As t changes sign, the motion of the electricity is reversed. *The motion of the electricity is therefore precisely the same in field* 3 *as in field* 1. Owing to the reversal of the sign of $\mathfrak{B}_x, \mathfrak{B}_y, \mathfrak{B}_z$, ($\mathfrak{E}_x, \mathfrak{E}_y, \mathfrak{E}_z$-retaining the same sign) the direction of the energy-stream is reversed, so that in field 3 we have a radiation of energy *inwards*. For a periodical motion of the electric charges the amount of energy drawn in during a period is the same as the radiation outwards in fields 1 and 2.

It is further easily found that field 3 may be calculated from *advanced* potentials, with a zero-contribution of infinity. If on the other hand the potentials are taken as delayed, the contribution of infinity is not equal to zero.

We now superpose field 1 and 3, which is possible since the field-equations are linear. In the two fields taken separately the electricity has the same motion, which therefore remains the same in the combined field. The density of the electricity on the other hand is everywhere doubled by the superposition. In order to reduce this to the former value we diminish the strength of the field in the combination to half its value. The density of the electricity is thereby diminished in the same ratio, and in the new *field* 4 we thus obtain precisely the same distribution and motion of the electricity as in field 1. The motion being periodical the energy-stream in field 4 evidently gives a total radiation zero during a full period. We have obtained a kind of stationary electro-magnetic waves.

Since the motion of the electricity in field 4 is identical with the motion from which we started, it has been proved that *every periodical motion of electric charges allows the assumption of an electromagnetic field without radiation of energy.* Without further enquiry it is clear, that this proposition also holds, if the motion of the charges is not exactly periodical, but consists in, say, a planetary motion with a movement of the perihelion.

The question remains, whether Bohr's theory can draw any benefit from the result arrived at, but it seems that such is not the case. If the electrons in an atom were going round in the same orbits for all eternity, there would be nothing to prevent us assuming an electro-magnetic field such as field 4. But the sudden transitions from one allowable orbit to another cause difficulties. A simple calculation shows that in field 4 the energy-density for large distances r is proportional to $\frac{1}{r^2}$, the energy of the whole field, therefore, becoming infinite. In consequence of this the change of energy associated with the transition of an electron has quite a different value to what Bohr has to assume, and it does not seem to me possible to make the two values agree. The above discussion, therefore, hardly seems to have a direct bearing on Bohr's theory, but it does seem to me to be of some use for obtaining a broader insight into the question as to where the difficulties of Bohr's theory actually lie. The result to which in my opinion it leads in this case was stated in the beginning of this note and I should like to formulate it in this way: the usual statement that it is inexplicable why an electron moving in accordance with the quantum-conditions should not radiate energy, seems to me to be based on an assumption which is not sufficiently general. A more general conception of the problem although unable to solve the difficulty, seems to me to put it in a different light.

Leiden, April 18, 1919.