Radiation reaction due to magnetic dipole radiation

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The radiation reaction due to magnetic dipole radiation is calculated for a charged-particle system whose magnetic moment changes with time. The explicit expressions for the electric and magnetic fields due to the magnetic dipole radiation reaction are derived, they are found to originate in the fourth-order term in the expansion of the retarded potential, whereas the field due to the electric dipole radiation reaction originates in the second-order term.

I. INTRODUCTION

The theory of radiation reaction occupies a unique position in the classical electromagnetism and the general theory of relativity. The famous example of the Lorentz frictional force due to electric dipole radiation has been fully treated in the textbook of Landau and Lifshitz on the classical theory of fields.[1] The radiation reaction due to gravitational wave radiation is of more recent interest, and it has been studied by many authors.[2-7]

The radiation reaction due to magnetic dipole radiation, on the other hand, has not been studied in detail despite its fundamental importance. In this paper we will derive the explicit expressions for the electric and magnetic fields due to magnetic dipole radiation reaction and show that they satisfy the necessary requirements for the radiation reaction fields. The present paper is organized as follows. In Sec. II we calculate the radiation reaction due to magnetic dipole radiation for a charged-particle system whose magnetic moment changes with time. Concluding remarks are given in Sec. III.

II. RADIATION REACTION FORCES

In this section we calculate the radiation reaction forces due to magnetic dipole radiation following Landau and Lifshitz.[1] We consider the field produced by a system of moving charges. We choose the origin of coordinates O anywhere in the interior of the system of charges. The radius vector from O to the point P, where we determine the field, is denoted by $\mathbf{R}_0$. Let the radius vector of the charge element $de=\rho\, dV$ be $\mathbf{r}$, and the radius vector from de to the point P be $\mathbf{R}$, thereby $\mathbf{R}=\mathbf{R}_0-\mathbf{r}$. Then the retarded potentials are given by

$$\phi(t,\mathbf{R}_0)=\int \frac{1}{R}\rho\left[t-\frac{R}{c},\mathbf{r}\right]dV\ , \tag{2.1}$$

$$\mathbf{A}(t,\mathbf{R}_0)=\frac{1}{c}\int \frac{1}{R}\mathbf{j}\left[t-\frac{R}{c},\mathbf{r}\right]dV\ , \tag{2.2}$$

where $\mathbf{j}$ is the electric current density of the charged-particle system. We expand the potentials as power series of R/c, assuming that R/c is small compared to the characteristic time of variation of the charged-particle system:

$$\begin{aligned}\phi(t,\mathbf{R}_0)=&\int \frac{1}{R}\rho(t,\mathbf{r})dV-\frac{1}{c}\frac{\partial}{\partial t}\int \rho(t,\mathbf{r})dV+\frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho(t,\mathbf{r})dV\\&-\frac{1}{6c^3}\frac{\partial^3}{\partial t^3}\int R^2\rho(t,\mathbf{r})dV+\frac{1}{24c^4}\frac{\partial^4}{\partial t^4}\int R^3\rho(t,\mathbf{r})dV-\frac{1}{120c^5}\frac{\partial^5}{\partial t^5}\int R^4\rho(t,\mathbf{r})dV\ ,\end{aligned} \tag{2.3}$$

$$\mathbf{A}(t,\mathbf{R}_0)=\frac{1}{c}\int \frac{1}{R}\mathbf{j}(t,\mathbf{r})dV-\frac{1}{c^2}\frac{\partial}{\partial t}\int \mathbf{j}(t,\mathbf{r})dV+\frac{1}{2c^3}\frac{\partial^2}{\partial t^2}\int R\,\mathbf{j}(t,\mathbf{r})dV-\frac{1}{6c^4}\frac{\partial^3}{\partial t^3}\int R^2\mathbf{j}(t,\mathbf{r})dV\ . \tag{2.4}$$

The second term in Eq. (2.3) vanishes because of charge conservation. Then we carry out the gauge transformation

$$\phi'=\phi-\frac{1}{c}\frac{\partial f}{\partial t}\ , \tag{2.5}$$

$$\mathbf{A}'=\mathbf{A}+\mathrm{grad} f\ , \tag{2.6}$$

$$f=\frac{1}{2c}\frac{\partial}{\partial t}\int R\rho(t,\mathbf{r})dV-\frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\int R^2\rho(t,\mathbf{r})dV+\frac{1}{24c^3}\frac{\partial^3}{\partial t^3}\int R^3\rho(t,\mathbf{r})dV-\frac{1}{120c^4}\frac{\partial^4}{\partial t^4}\int R^4\rho(t,\mathbf{r})dV\ . \tag{2.7}$$

Then we obtain

$$\phi'=\int \frac{1}{R}\rho(t,\mathbf{r})dV\equiv\phi'^{(0)} , \tag{2.8}$$

$$\begin{aligned}\mathbf{A}'=&\frac{1}{c}\int \frac{1}{R}\mathbf{j}(t,\mathbf{r})dV+\frac{1}{2c}\frac{\partial}{\partial t}\mathrm{grad}\int R\rho(t,\mathbf{r})dV-\frac{1}{c^2}\frac{\partial}{\partial t}\int \mathbf{j}(t,\mathbf{r})dV-\frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\mathrm{grad}\int R^2\rho(t,\mathbf{r})dV\\ &+\frac{1}{2c^3}\frac{\partial^2}{\partial t^2}\int R\,\mathbf{j}(t,\mathbf{r})dV+\frac{1}{24c^3}\frac{\partial^3}{\partial t^3}\mathrm{grad}\int R^3\rho(t,\mathbf{r})dV-\frac{1}{6c^4}\frac{\partial^3}{\partial t^3}\int R^2\mathbf{j}(t,\mathbf{r})dV-\frac{1}{120c^4}\frac{\partial^4}{\partial t^4}\mathrm{grad}\int R^4\rho(t,\mathbf{r})dV\\ \equiv&\ \mathbf{A}'^{(1)}+\mathbf{A}'^{(2)}+A'^{(3)}+\mathbf{A}'^{(4)} .\end{aligned} \tag{2.9}$$

The second-order term $\mathbf{A}'^{(2)}$ [the third and fourth terms in Eq. (2.9)] gives rise to the radiation reaction potential due to electric dipole radiation:[1]

$$\mathbf{A}'^{(2)}=-\frac{2}{3c^2}\ddot{\mathbf{d}} , \tag{2.10}$$

where $\mathbf{d}$ is the electric dipole moment of the charged-particle system. The corresponding electric field is given by

$$\mathbf{E}^{(2)}=-\frac{1}{c}\dot{\mathbf{A}}'^{(2)}=\frac{2}{3c^3}\dddot{\mathbf{d}} . \tag{2.11}$$

The corresponding magnetic field $\mathbf{B}^{(2)}$ vanishes in this case.

The fourth-order term $\mathbf{A}'^{(4)}$ is written as

$$\mathbf{A}'^{(4)}=\frac{1}{15c^4}\frac{\partial^3}{\partial t^3}\sum e[-2R^2\mathbf{v}+(\mathbf{R}\cdot\mathbf{v})\mathbf{R}] , \tag{2.12}$$

where the summation is carried out over all charges. The corresponding magnetic field is calculated as

$$\mathbf{B}^{(4)}=\mathrm{curl}\,\mathbf{A}'^{(4)}=-\frac{1}{3c^4}\mathbf{R}_0\times\dddot{\mathbf{d}}+\frac{2}{3c^3}\dddot{\mathbf{m}} , \tag{2.13}$$

where $\mathbf{m}$ is the magnetic moment of the charged-particle system. The electric field is given by

$$\begin{aligned}\mathbf{E}^{(4)}=&-\frac{1}{c}\frac{\partial\mathbf{A}'^{(4)}}{\partial t}\\ =&\frac{1}{3c^4}\mathbf{R}_0\times\ddddot{\mathbf{m}}+\frac{2}{15}\frac{R_0^2}{c^5}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathbf{d}}-\frac{1}{30}\frac{R_0}{c^5}\overset{\cdot\cdot\cdot\cdot\cdot}{\mathbf{D}}\\ &-\frac{1}{15c^5}\mathbf{R}_0(\mathbf{R}_0\cdot\overset{\cdot\cdot\cdot\cdot\cdot}{\mathbf{d}})+\frac{1}{6c^5}\frac{\partial^4}{\partial t^4}\sum e\mathbf{v}(\mathbf{r}\cdot\mathbf{r})\\ &-\frac{1}{30c^5}\frac{\partial^5}{\partial t^5}\sum e\mathbf{r}(\mathbf{r}\cdot\mathbf{r}) ,\end{aligned} \tag{2.14}$$

where $\mathbf{D}$ is the vector with components $D_\alpha=D_{\alpha\beta}n_\beta$

$$\mathbf{D}=\sum e[3\mathbf{r}(\mathbf{n}\cdot\mathbf{r})-\mathbf{n}r^2] , \tag{2.15}$$

$D_{\alpha\beta}$ being the electric quadrupole moment and $\mathbf{n}$ being $\mathbf{n}=\mathbf{R}_0/R_0$.

For a charged-particle system with $\mathbf{d}=0$, $D_{\alpha\beta}=0$, $\sum e\mathbf{v}(\mathbf{r}\cdot\mathbf{r})=0$, $\sum e\mathbf{r}(\mathbf{r}\cdot\mathbf{r})=0$, we have

$$\mathbf{B}^{(4)}=\frac{2}{3c^3}\dddot{\mathbf{m}} , \tag{2.16}$$

$$\mathbf{E}^{(4)}=\frac{1}{3c^4}\mathbf{R}_0\times\ddddot{\mathbf{m}} . \tag{2.17}$$

The force due to these fields acting on each charge e is

$$\mathbf{f}=e\left[\mathbf{E}^{(4)}+\frac{\mathbf{v}}{c}\times\mathbf{B}^{(4)}\right] . \tag{2.18}$$

The work done by this force in unit time is $\mathbf{f}\cdot\mathbf{v}$, so that the averaged value of the total work performed on all the charges is equal to the sum, taken over all the charges:

$$\begin{aligned}\overline{\sum \mathbf{f}\cdot\mathbf{v}}=&\overline{\int \mathbf{j}\cdot\mathbf{E}^{(4)}dV}\\ =&\overline{\sum\frac{e}{3c^4}(\mathbf{r}\times\ddddot{\mathbf{m}})\cdot\mathbf{v}}\\ =&-\frac{2}{3c^3}\overline{\mathbf{m}\cdot\ddddot{\mathbf{m}}}=-\frac{2}{3c^3}\overline{\ddot{\mathbf{m}}^2} .\end{aligned} \tag{2.19}$$

The decrease in angular momentum per unit time $d\mathbf{M}/dt$ due to this force is calculated as

$$\begin{aligned}\frac{d\mathbf{M}}{dt}=&\sum \mathbf{r}\times\mathbf{f}\\ =&\sum\frac{e}{3c^4}\frac{d}{dt}[\mathbf{r}\times(\mathbf{r}\times\dddot{\mathbf{m}})]+\frac{2}{3c^3}\mathbf{m}\times\dddot{\mathbf{m}} ,\end{aligned} \tag{2.20}$$

Considering the time average of the loss of angular momentum for a stationary motion, we obtain

$$\dot{\overline{\mathbf{M}}}=\frac{1}{2c}\overline{\int(\mathbf{r}\times\mathbf{j})\times\mathbf{B}^{(4)}dV}=\frac{2}{3c^3}\overline{\mathbf{m}\times\dddot{\mathbf{m}}}=-\frac{2}{3c^3}\overline{\dot{\mathbf{m}}\times\ddot{\mathbf{m}}} . \tag{2.21}$$

It should be remarked that Eqs. (2.19) and (2.21) are fully consistent with the loss of energy and angular momentum due to magnetic dipole radiation calculated for the wave zone of the radiation.[1] Hence we have shown that the fields in Eqs (2.16) and (2.17) are radiation reaction fields for magnetic dipole radiation. This situation is completely analogous to the case of electric dipole radiation.[1]

III. CONCLUDING REMARKS

We have derived the explicit expressions for the electric and magnetic fields due to magnetic dipole radiation reaction. These fields have been shown to originate in the fourth-order term in the expansion of the retarded potential, whereas the field due to electric dipole radiation reaction originates in the second-order term. We have verified that the fields so obtained satisfy the necessary requirements for the radiation reaction fields. The derived electric and magnetic fields due to magnetic dipole radiation reaction have been shown to have much in common with the field due to electric dipole radiation reaction.

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