The problem of spherically symmetric electromagnetic radiation

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Using Maxwell equations for the radiation's transverse fields, it is proved that a genuine spherically symmetric electromagnetic radiation cannot exist.
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The existence of a spherically symmetric wave might be regarded as a self-evident phenomenon having a rather simple mathematical structure. Such a structure is obviously true for the longitudinal sound wave emitted from a breathing sphere that is placed in an isotropic medium. Also, the blackbody radiation emitted from an isothermic sphere is statistically spherical. These kinds of waves are not discussed here. We will discuss waves in the wave zone whose distance from the source is much greater than both the source's linear size and the radiation's wavelength. The problem analyzed here is called "genuine spherically symmetric electromagnetic radiation." This term means that *at the wave zone*, there exists a set of concentric spherical shells, where at all points of each shell, the Poynting vector[1,2]

$$\mathbf{S}=\frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \tag{1}$$

is *radial* and has the *same* magnitude. (This magnitude varies as the radius of the shell changes and may vary as a function of time.)

The case analyzed in this paper is not restricted to states whose fields have a well-defined parity. Indeed, the Poynting vector has a well-defined parity $\mathbf{S}(\mathbf{r})=-\mathbf{S}(-\mathbf{r})$. On the other hand, the transverse electromagnetic fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are not necessarily related to $\mathbf{E}(-\mathbf{r})$ and $\mathbf{B}(-\mathbf{r})$, respectively.

Unlike longitudinal sound waves, a system emitting a genuine spherically symmetric electromagnetic radiation is far from being trivial. Indeed, an elementary radiating object is either a quantum mechanical system emitting appropriate dipole or higher multipole radiation or a loop of a time-dependent classical current. The latter can also be expanded in multipoles.[3] As is well known, each of these objects emits radiation whose intensity varies with direction. However, it is not clear whether or not one can arrange infinitely many multipoles (each of which emits an infinitesimal amount of energy) so that the entire result takes the form of a genuine spherically symmetric electromagnetic radiation.

As is well known, electrodynamics examines two kinds of entities, charges and their currents on one hand and electromagnetic fields on the other. An examination of the possible ways of building a radiating system containing an arbitrary number of dipoles (or higher multipoles) looks very difficult. In the following, the fields of a genuine spherically symmetric radiation are analyzed, and it is shown that this kind of radiation is inconsistent with Maxwell equations. This conclusion enhances the insight into the internal structure of electrodynamics. The proof assumes that a genuine spherically symmetric electromagnetic radiation exists and arrives at a contradiction. The case discussed here provides an example of constraints imposed by Maxwell equations, and indicates that not every arrangement of electromagnetic fields can be realized.

Let us consider a spherical shell S belonging to the set mentioned above and examine it at an instant when $\mathbf{S}$ in Eq. (1) does not vanish. Because the electromagnetic fields are perpendicular to $\mathbf{S}$, they have no radial component. Moreover, because at all points on S, the magnitude of the Poynting vector is the same, the magnitude of the transverse field $\mathbf{E}$ (and $\mathbf{B}$) is the same also. The latter conclusion relies on the orthogonality of $\mathbf{E}$ and $\mathbf{B}$ at the wave zone.[4]

Consider a point P on the spherical shell S. $\mathbf{E}(P)$ denotes the electric field at P. The direction of $\mathbf{E}(P)$ determines a great circle on S that passes through P such that the electric field $\mathbf{E}(P)$ is tangent to it. Henceforth, this great circle is called the equator (see Fig. 1).

Now let us construct a trajectory C on S that is based on the following assumptions. C starts at P. Also for every point p on C, the tangent to C at p is in the direction of $\mathbf{E}(p)$. Because $|\mathbf{E}|$ has the same value at all points of S, $\mathbf{E}$ does not vanish there and the definition of the tangent is a unique function defined at every point belonging to S. This definition of C is unique. Indeed, the tangent unit vector of differential geometry takes the following form:

$$\mathbf{t}\equiv\frac{d\mathbf{r}}{dl}=\frac{\mathbf{E}}{E}, \tag{2}$$

where l denotes the arclength.[5] Hence, we obtain a well-defined ordinary first-order differential equation. Thus, by the existence and uniqueness theorems of differential equations,[6] C is well-defined and unique.

The trajectory C is used below and the following cases are analyzed:

(1) One can find a quantity $\epsilon>0$ such that the trajectory C passes at the point Q whose distance from the equator is greater than ϵ (see Fig. 1).
(2) Otherwise.

Assume that case (1) holds. Thus after reaching point Q, the trajectory C is closed by adding to it the shorter part C' of the great circle that passes through P and Q (the arc QRP in Fig. 1). Evidently, the length of C' is shorter than that of C. The above assumptions imply that the value of the following path integral is positive,

$$\oint_{C+C'}\mathbf{E}\cdot d\mathbf{l}=\int_{C}\mathbf{E}\cdot d\mathbf{l}+\int_{C'}\mathbf{E}\cdot d\mathbf{l}>0. \tag{3}$$

Indeed, the length of C is greater than that of C' and, at every point of C, the electric field $\mathbf{E}$ is tangent to the path.

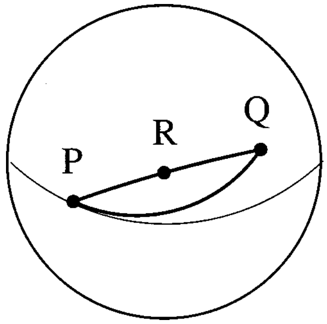

Fig. 1. A sphere at the wave zone. The thin line denotes a great circle called the equator. The path *PQ* is tangent to the equator at point *P*. The path *QRP* is a portion of a great circle (see the text).

Thus, $\cos\theta$ of the scalar product takes the maximal value of unity. Because $|\mathbf{E}|$ is uniform on S, the integral on C is greater than the absolute value of that of C'.

Now, let us turn to the second case. Here C coincides with the equator which is taken as the closed path. Hence, in this case C' reduces to a point, but Eq. (3) still holds. Using vector analysis, we find from Eq. (3) that

$$\oint_{C+C'} \mathbf{E}\cdot d\mathbf{l} = \int_{S'} (\boldsymbol{\nabla}\times\mathbf{E})\cdot d\mathbf{s} > 0, \tag{4}$$

where S' is any surface whose boundary is the closed path $C+C'$. Let us choose the corresponding part of the spherical shell S as the surface S'. It follows from Eq. (4) that there exists a region on S' where the radial part of the following vector satisfies $(\boldsymbol{\nabla}\times\mathbf{E})_r > 0$. Hence, Maxwell's equation yields

$$(\boldsymbol{\nabla}\times\mathbf{E})_r = -\frac{1}{c}\frac{\partial B_r}{\partial t} > 0. \tag{5}$$

Thus, the assumption that genuine spherically symmetric radiation exists yields a contradiction, because for any spherical radiation, the longitudinal field component B_r together with its derivative with respect to the time, vanishes identically.[7] This result completes the proof that a genuine spherically symmetric electromagnetic radiation cannot exist. It can be easily seen that the proof holds not only for an outgoing radiation but for an incoming one as well.

As stated, there is an infinite number of different systems emitting electromagnetic radiation. Each of these systems is a particular arrangement of electric and magnetic dipoles or higher multipoles and appropriate classical currents. Thus, there are infinitely many patterns of electromagnetic radiation. However, in spite of the infinite degrees of freedom available, not every pattern of radiation can be realized.

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[1]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975), p. 76.
[2]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), p. 237.
[3]See Ref. 2, pp. 744–747.
[4]See Ref. 1, p. 162, and Ref. 2, p. 657.
[5]D. E. Rutherford, *Vector Methods* (Oliver and Boyd, Edinburgh, 1962), 9th ed., pp. 16–19.
[6]E. L. Ince, *Ordinary Differential Equation* (Dover, New York, 1956), Chap. III.
[7]See Ref. 1, p. 162, Ref. 2, p. 657.

NOTES AND DISCUSSIONS

Comment on "The problem of spherically symmetric electromagnetic radiation," by E. Comay [Am. J. Phys. 70 (7), 715–716 (2002)]

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E. Comay has recently pointed out that Maxwell's equations forbid the existence of a spherically symmetric electromagnetic radiation field.[1] Although it has long been known that the emission of radiation is anisotropic (Einstein used this fact to motivate the photon picture[2]), Comay points out that no possible arrangement of multiple emitters can produce a spherical wave. It turns out that Maxwell's equations are not needed to prove this result. The result is a simple consequence of the mathematical theorem that there exists no continuous unit tangent vector fields defined over a spherical surface.[3] As examples, the gradient of neither the latitude nor longitude is defined at the poles. Similarly, there are no non-singular tangential electric or magnetic fields defined over a whole spherical surface, as would be required for a radiation field. Another physical consequence of this theorem is that there is no spherically symmetric distribution of gravitational radiation that is everywhere nonzero. This is necessarily the case regardless of the detailed form of the gravitational theory, requiring only that this radiation be transverse, $h_\nu k^\nu = 0$, where h_ν is the field strength and k^ν is the wave four-vector. This result follows because we can always construct a tangent vector to the wave front from h_ν by projection.

The reader can no doubt think of other interesting applications of this theorem.

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[1]E. Comay, "The problem of spherically symmetric electromagnetic radiation," Am. J. Phys. **70**, 715–716 (2002).
[2]A. Einstein, "Zur Quantentheorie der Strahlung," Z. Physik **18**, 121–128 (1917). English translation by H. A. Boorse and L. Motz, in *The World of the Atom* (Basic Books, New York, 1966), pp. 188–901.
[3]P. J. Hilton and S. Wylie, *Homology Theory* (Cambridge U.P., Cambridge, 1960), p. 219.

Comment on "The problem of spherically symmetric electromagnetic radiation," by E. Comay [Am. J. Phys. 70 (7), 715–716 (2002)]

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The recent paper by Comay[1] on the subject of the impossibility of spherically symmetric electromagnetic radiation did not reference the original work on the subject, of which there are at least two to my knowledge.

That a system of coherent currents radiating isotropically in all directions of a sphere is a physical impossibility was first proved by Mathis[2] in 1951 using a theorem due to Brouwer.[3] This theorem states that a continuous vector function $\mathbf{A}(\theta,\phi)$ on a sphere that has the property that $\mathbf{A}\cdot\mathbf{r}=0$ with $A\neq 0$ is impossible for all (θ,ϕ).

The other proof is due to Boukamp and Casimir.[4] In this classic paper they first deduced the multipole expansion of the radiation field as exact solutions of Maxwell's equations.[5] These expressions are

$$\mathbf{E}=\frac{e^{ikr}}{r}\left[\sum_{\ell m} a_{\ell m}(-\imath)^{\ell+1}\mathbf{e}_r\times\mathbf{L}Y_\ell^m + \sum_{\ell m} b_{\ell m}\sqrt{\frac{\mu}{\epsilon}}(-\imath)^{\ell+1}\mathbf{L}Y_\ell^m\right],$$

$$\mathbf{B}=\frac{e^{ikr}}{r}\left[-\sum_{\ell m} a_{\ell m}\sqrt{\frac{\mu}{\epsilon}}(-\imath)^{\ell+1}\mathbf{L}Y_\ell^m + \sum_{\ell m} b_{\ell m}(-\imath)^{\ell+1}\mathbf{e}_r\times\mathbf{L}Y_\ell^m\right],$$

in which the multipole coefficients are given in terms of integrals over the source current distributions:

$$a_{\ell m} = -\frac{1}{4\pi}\sqrt{\frac{\mu}{\epsilon}}\frac{(-1)^m}{\ell(\ell+1)}\int_{V_0}\mathbf{J}(\mathbf{r}')\cdot\nabla'\times\nabla'$$

$$\times(\mathbf{r}' j_\ell(kr)Y_\ell^{-m})dV',$$

$$b_{\ell m} = \frac{\imath k}{4\pi}\frac{(-1)^m}{\ell(\ell+1)}\int_{V_0}\mathbf{J}(\mathbf{r}')\cdot\nabla'\times(\mathbf{r}' j_\ell(kr)Y_\ell^{-m})dV'.$$

For arbitrary current distributions one can show[4] from these expressions that an isotropic radiation pattern is impossible. However in the long wavelength limit, $ka \ll 1$, for atomic radiators where the multipole coefficients (properly calculated using quantum mechanics) are independent of m,[5] the radiation is isotropic. Once again, quantum mechanics breaks the ''classical'' rules, a situation not unlike the violation of the classical Bohr–van Leeuven theorem of diamagnetism.

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[1]E. Comay, ''The problem of spherically symmetric electromagnetic radiation,'' Am. J. Phys. **70** (7), 715–716 (2002).
[2]H. F. Mathis, ''A Short Proof that an Isotropic Antenna Is Impossible,'' Proc. IRE **39**, 970 (1951).
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Reply to ''Comments on 'The problem of spherically symmetric electromagnetic radiation,' '' by E. Comay [Am. J. Phys. 70 (7), 715–716 (2002)]

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The comments[1,2] by Rosenthal and Choy give much insight into the nature of the restrictions imposed on vector fields. They correctly show that a transverse vector field cannot have a uniform size over an entire spherical shell. [The claim holds also for a nonvanishing vector field, because such a field can be cast into a unit vector field by means of a multiplication factor $f(\theta,\varphi)$.] Thus, the use of Maxwell's equations looks redundant.

However, we may use Maxwell's equations for the transverse electromagnetic fields and take an additional step. The proof given in Ref. 3 holds for the spherical region R defined at the equator between latitudes $\pm d$. The proof requires that the electric field at a point P on the equator is tangent to it. This requirement can be satisfied by applying a duality transformation to the fields,[4] $\mathbf{E}' = \mathbf{E}\cos\alpha + \mathbf{B}\sin\alpha$ and $\mathbf{B}' = -\mathbf{E}\sin\alpha + \mathbf{B}\cos\alpha$, and fixing the value of α.

If a path C departs from the equator, then the proof in Ref. 3 holds [see case (1), a few lines after Eq. (2) therein]. If, on the other hand, the path C coincides with the equator, then one makes a closed path using the following sections: a section S of a longitude between angles $0 \leqslant \theta \leqslant d'' < d$, and a full circle C'' on the latitude d''. The integral on the closed path $C - S - C'' - S$ can be used for completing the proof.

Thus, electromagnetic fields cannot take a uniform magnitude in the spherical region R defined above. This property is not satisfied by a general transverse vector field, as shown by the following example: $\mathbf{v}(\theta,\varphi) = \mathbf{u}_\varphi$ $(-d \leqslant \theta \leqslant d, 0 \leqslant \varphi < 2\pi)$, where $\mathbf{u}_\varphi$ is the ordinary unit vector in spherical coordinates.

As correctly pointed out by Choy,[2] the discussion in Ref. 3 is restricted to classical physics. Indeed, the probability function of a photon emitted from an atom whose state is m-fold degenerate is uniform on a sphere.

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[1]A. S. Rosenthal, ''Comment on 'The problem of spherically symmetric electromagnetic radiation,' '' by E. Comay [Am. J. Phys. **70** (7), 715–716 (2002)], Am. J. Phys. **71**, 91 (2003).
[2]T. C. Choy, ''Comment on 'The problem of spherically symmetric electromagnetic radiation,' '' by E. Comay [Am. J. Phys. **70** (7), 715–716 (2002)], Am. J. Phys. **71**, 91 (2003).
[3]E. Comay, ''The problem of spherically symmetric electromagnetic radiation,'' Am. J. Phys. **70**(7), 715–716 (2002).
[4]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1998), p. 274.

