

# Linear Momentum of the Source of a Static Electromagnetic Field\*

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*It is shown that in general a stationary system of charges and currents generating a static electromagnetic field has net linear momentum. Explicit expressions for this momentum are found, and the physical origin of the momentum is brought out by means of examples.*

## INTRODUCTION

Within the past few years it has been discovered that a correct discussion of the linear momentum of a system of stationary charges and currents and their associated electric and magnetic fields is somewhat more subtle than one might expect. One's intuition would probably say that the net linear momentum is zero, and this is indeed correct. However, perhaps contrary to one's intuition, the linear momentum of each of the subsystems "field" and "source" is in general not zero. In a previous article<sup>1</sup> the author discussed the linear momentum of a quasistatic electromagnetic field. In this article, following the lead of Shockley and James,<sup>2</sup> of Penfield and Haus,<sup>3</sup> and of others,<sup>4</sup> the linear momentum of the source of the field is discussed.

### I. A GENERAL RESULT

We begin by showing that the net linear momentum of any closed stationary system is zero. Consider a system in which there is a total energy flux  $\mathbf{S}$  and hence<sup>5</sup> a momentum density  $\mathbf{S}/c^2$ . The net linear momentum of the system is

$$\mathbf{P} = (1/c^2) \int d^3\mathbf{x}\mathbf{S}. \tag{1}$$

By using the vector identity

$$\int d^3\mathbf{x}\mathbf{S} = \oint d\mathbf{a}\mathbf{x}\mathbf{n} \cdot \mathbf{S} - \int d^3\mathbf{x}\mathbf{x}\nabla \cdot \mathbf{S}, \tag{2}$$

one can rewrite Eq. (1) as

$$\mathbf{P} = (1/c^2) \oint d\mathbf{a}\mathbf{x}\mathbf{n} \cdot \mathbf{S} - (1/c^2) \int d^3\mathbf{x}\mathbf{x}\nabla \cdot \mathbf{S}. \tag{3}$$

The contribution of the surface integral vanishes for a closed system since  $\mathbf{n} \cdot \mathbf{S}$  either vanishes on the boundary for a finite system or tends to zero sufficiently fast at infinity for an infinite system. Further, the energy flux  $\mathbf{S}$  and the energy density  $u$  are related by the equation of continuity

$$\dot{u} + \nabla \cdot \mathbf{S} = 0. \tag{4}$$

Thus, the net linear momentum of a closed system can be written<sup>6</sup>

$$\mathbf{P} = \int d^3\mathbf{x}\mathbf{x}(\dot{u}/c^2). \tag{5}$$

Equation (5) shows in particular that the net linear momentum of a closed stationary system (for which  $\dot{u}$  is zero) is zero.

Now apply this result to a system consisting of stationary charges and currents and their associated electric and magnetic fields  $\mathbf{E}$  and  $\mathbf{B}$ . The energy flux of the electromagnetic field is given by Poynting's vector  $\mu_0^{-1}\mathbf{E} \times \mathbf{B}$ , the momentum density by  $\epsilon_0\mathbf{E} \times \mathbf{B}$ . Hence, the linear momentum of the electromagnetic field is

$$\mathbf{P}_{EM} = \int d^3\mathbf{x}\epsilon_0\mathbf{E} \times \mathbf{B}. \tag{6}$$

It is useful to rewrite this by setting  $\mathbf{E} = -\nabla\phi$  with  $\phi$  the electrostatic potential, integrating by parts with the aid of the identity  $\nabla \times (\phi\mathbf{B}) = \nabla\phi \times \mathbf{B} + \phi\nabla \times \mathbf{B}$ , and using the field equation  $\nabla \times \mathbf{B} = \mu_0\mathbf{J}$  with  $\mathbf{J}$  the electric current density. One finds

$$\mathbf{P}_{EM} = (1/c^2) \int d^3\mathbf{x}\phi\mathbf{J}. \tag{7}$$

Yet another useful expression is obtained by setting  $\phi = (1/4\pi\epsilon_0) \int d^3\mathbf{x}'\rho/r$  with  $\rho$  the electric charge density, interchanging the order of integration over  $\mathbf{x}'$  and  $\mathbf{x}$ , and remembering that the transverse vector potential  $\mathbf{A}^T$  is given by  $\mathbf{A}^T = (\mu_0/4\pi) \int d^3\mathbf{x}\mathbf{J}/r$ .

One finds

$$\mathbf{P}_{EM} = \int d^3\mathbf{x} \rho \mathbf{A}^T. \quad (8)$$

The necessity for linear momentum in a static electromagnetic field, and its representation by Eq. (8) has been discussed previously.<sup>1</sup>

The linear momentum  $\mathbf{P}_{EM}$  in a static electromagnetic field is in general nonzero.<sup>7</sup> Thus, in order for the total momentum  $\mathbf{P}$  of the system to be zero, the charges and currents generating the fields must have linear momentum  $\mathbf{P}_{ME}$  of magnitude

$$\mathbf{P}_{ME} = -\mathbf{P}_{EM}. \quad (9)$$

Using Eqs. (6)–(8), one can write the explicit expressions

$$\mathbf{P}_{ME} = - \int d^3\mathbf{x} \epsilon_0 \mathbf{E} \times \mathbf{B}, \quad (10)$$

$$= - (1/c^2) \int d^3\mathbf{x} \phi \mathbf{J}, \quad (11)$$

$$= - \int d^3\mathbf{x} \rho \mathbf{A}^T. \quad (12)$$

## II. EXAMPLES

Although the preceding argument is mathematically simple and general, it offers no insight into the physical mechanism whereby a stationary system of charges and currents can have net linear momentum. This is best brought out by means of examples.

A particularly simple illustration<sup>2,3</sup> and yet one which contains the essential features of more complicated systems is the following: Consider a particle moving around a closed frictionless track. The instantaneous linear momentum is  $\mathbf{P} = m\mathbf{v}$  with  $m = m_0(1 - v^2/c^2)^{-1/2}$  the relativistic mass and  $\mathbf{v}$  the instantaneous velocity. The time per cycle the particle spends on an element of track  $dl$  is  $dl/v$ , and thus the time average momentum is

$$\langle \mathbf{P} \rangle_{av} = T^{-1} \oint m dl, \quad (13)$$

where  $T = \oint dl/v$  is the period of the motion. In nonrelativistic mechanics,  $m$  is constant and  $\langle \mathbf{P} \rangle_{av}$  is zero. In relativistic mechanics, where  $m$  varies with  $v$ , this is not necessarily the case. If the particle carries a charge  $q$ , and the track is situated in a region in which there is an electro-

static field described by a potential  $\phi$ , the energy conservation equation reads

$$mc^2 + q\phi = \mathcal{E}, \quad (14)$$

with  $\mathcal{E}$  the constant total energy of the particle. Using this equation to eliminate the relativistic mass  $m$  from Eq. (13), one finds

$$\langle \mathbf{P} \rangle_{av} = - \langle \langle I \rangle_{av} / c^2 \rangle \oint \phi dl \quad (15)$$

where  $\langle I \rangle_{av} = q/T$  is the time average electric current on the track. Equation (15) is just a special case of Eq. (11). The origin of the nonzero average momentum is clear: It is due to the fact that the momentum and velocity of the particle are not proportional to one another.

It is of interest to consider the case in which the dimensions of the orbit are small in comparison with the distance over which the potential changes appreciably. In this case one can expand  $\phi(\mathbf{x})$  in a Taylor series about a point  $\mathbf{x}_0$  fixed near the center of the orbit and retain only the first few terms in the series

$$\phi(\mathbf{x}) = \phi(\mathbf{x}_0) - (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{E}(\mathbf{x}_0) + \dots \quad (16)$$

Substituting this expansion into Eq. (15) and using the identity

$$\oint x_i dx_j = \epsilon_{ijk} A_k \quad (17)$$

(with  $\epsilon_{ijk} = +1$  if  $ijk$  is an even permutation of 123,  $-1$  if  $ijk$  is an odd permutation, and zero otherwise, and  $A_k$  is the area of the projection of the orbit onto a plane normal to the  $k$  axis), one finds for the time average linear momentum

$$\langle \mathbf{P} \rangle_{av} \approx - (1/c^2) \mathbf{E} \times \mathbf{m}, \quad (18)$$

where  $\mathbf{m} = \langle I \rangle_{av} \mathbf{A}$  is the magnetic dipole moment of the system. This expression appears frequently in other works,<sup>2-4</sup> where it is applied to a small nonconducting magnet in an electric field.

We now turn to a more sophisticated although basically no more complicated example. Consider a stationary relativistic gas of point particles of charge  $q$  and rest mass  $m_0$  moving in a region and generating a static electromagnetic field described by potentials  $\phi$  and  $\mathbf{A}$ . One can with little difficulty

generalize this model to apply to gases containing more than one species of particle. Suppose that the gas is described by a distribution function  $f(\mathbf{x}, \mathbf{P})$  with  $f(\mathbf{x}, \mathbf{P}) d^3\mathbf{x} d^3\mathbf{P}$  the number of particles in the volume  $d^3\mathbf{x}$  centered on  $\mathbf{x}$  with *canonical* momentum in the range  $d^3\mathbf{P}$  centered on  $\mathbf{P}$ . The canonical momentum  $\mathbf{P}$  is related to the kinematic or ordinary momentum  $m\mathbf{v}$  by the equation

$$\mathbf{P} = m\mathbf{v} + q\mathbf{A}. \quad (19)$$

The distribution function  $f$  satisfies Boltzmann's equation

$$[f, H] = \nabla_x f \cdot \nabla_P H - \nabla_P f \cdot \nabla_x H = 0, \quad (20)$$

with  $H = c[(\mathbf{P} - q\mathbf{A})^2 + m_0^2 c^2]^{1/2} + q\phi$  the one-particle Hamiltonian. The kinematic linear momentum density of the particles is

$$\int d^3\mathbf{P} (\mathbf{P} - q\mathbf{A}) f. \quad (21)$$

With the aid of the identity

$$\mathbf{P} - q\mathbf{A} = (1/c^2) (H - q\phi) \mathbf{v}, \quad (22)$$

one can rewrite formula (21) as

$$\int d^3\mathbf{P} (H/c^2) \mathbf{v} f - (1/c^2) \phi \mathbf{J}, \quad (23)$$

with  $\mathbf{J} = \int d^3\mathbf{P} q \mathbf{v} f$  the electric current density. The divergence of the first term in formula (23) is zero<sup>8</sup> as a consequence of Boltzmann's equation (20) and the relation  $\mathbf{v} = \nabla_P H$ . Further, this term is zero outside the system. Thus, according to vector identity (2), when one integrates formula (23) for the linear momentum density of the particles over the system to obtain the total linear momentum of the particles, the first term gives zero contribution and one finds

$$\mathbf{P}_{ME} = - (1/c^2) \int d^3\mathbf{x} \phi \mathbf{J}, \quad (24)$$

in agreement with Eq. (11).

One would expect analogous results to hold in relativistic quantum mechanics. Indeed, they must hold if the general result of Sec. I is not violated. To show the explicit form the results

take, consider an electron with energy  $\varepsilon$  moving in an electromagnetic field described by a scalar potential  $\phi$  and vector potential  $\mathbf{A}$ . The behavior of the electron is described by a four component state function  $\psi$ , which satisfies Dirac's equation<sup>9</sup>

$$(1/c) (\varepsilon - e\phi) \psi = \boldsymbol{\alpha} \cdot (-i\hbar \nabla - e\mathbf{A}) \psi + m_0 c \beta \psi. \quad (25)$$

The quantities  $\alpha_i$  and  $\beta$  are  $4 \times 4$  Hermitian matrices which satisfy the commutation relations

$$\begin{aligned} \alpha_i \alpha_j + \alpha_j \alpha_i &= 2\delta_{ij} I, \\ \alpha_i \beta + \beta \alpha_i &= 0, \\ \beta^2 &= I, \end{aligned} \quad (26)$$

with  $\delta_{ij}$  the Kronecker  $\delta$ :  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  if  $i \neq j$ .  $I$  is the  $4 \times 4$  unit matrix. A common representation for the matrices is

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (27)$$

with

$$\begin{aligned} \sigma_1 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \sigma_2 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \\ \sigma_3 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \end{aligned} \quad (28)$$

the Pauli spin matrices.

The Hermitian adjoint state function  $\psi^\dagger$  satisfies the equation

$$(1/c) (\varepsilon - e\phi) \psi^\dagger = (i\hbar \nabla - e\mathbf{A}) \psi^\dagger \cdot \boldsymbol{\alpha} + m_0 c \psi^\dagger \beta. \quad (29)$$

If one multiplies Eq. (25) by  $\psi^\dagger \boldsymbol{\alpha}$  on the left, and Eq. (29) by  $\boldsymbol{\alpha} \psi$  on the right, and adds the two resulting equations, one finds

$$\begin{aligned} (2/c^2) (\varepsilon - e\phi) \psi^\dagger \boldsymbol{\alpha} \psi &= \psi^\dagger \boldsymbol{\alpha} \cdot [(-i\hbar \nabla - e\mathbf{A}) \psi] \\ &+ [(i\hbar \nabla - e\mathbf{A}) \psi^\dagger] \cdot \boldsymbol{\alpha} \boldsymbol{\alpha} \psi + m_0 c \psi^\dagger (\boldsymbol{\alpha} \beta + \beta \boldsymbol{\alpha}) \psi. \end{aligned} \quad (30)$$

The third term on the right-hand side of Eq. (30) is zero on account of the commutation relations (26). To simplify the first two terms on the

right-hand side of Eq. (30), consider the  $i$ th component

$$\psi^+ \alpha_i \alpha_j [(-i\hbar \partial / \partial x_j - eA_j) \psi] + [(i\hbar \partial / \partial x_j - eA_j) \psi^+] \alpha_j \alpha_i \psi. \quad (31)$$

Using the commutation relations (26) and the definition

$$\sigma_i' = -\frac{1}{2} i \epsilon_{ijk} \alpha_j \alpha_k, \quad (32)$$

set

$$\alpha_i \alpha_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k'. \quad (33)$$

Note that  $\frac{1}{2} \hbar \sigma'$  is the spin angular momentum operator, and in the representation (27) of the  $\alpha_i$  and  $\beta$  matrices,  $\sigma'$  is given by

$$\sigma' = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}. \quad (34)$$

Using Eq. (33) one can write the right-hand side of Eq. (30) as

$$\psi^+ [(-i\hbar \partial / \partial x_i - eA_i) \psi] + [(i\hbar \partial / \partial x_i - eA_i) \psi^+] \times \psi^+ \epsilon_{ijk} (\partial / \partial x_j) [\psi^+ \hbar \sigma_k \psi]. \quad (35)$$

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<sup>1</sup> M. G. Calkin, Amer. J. Phys. **34**, 921 (1966); see also the related articles of R. H. Romer, Amer. J. Phys. **34**, 772 (1966); **35**, 445 (1967), and of E. M. Pugh and G. E. Pugh, Amer. J. Phys. **35**, 153 (1967).

<sup>2</sup> W. Shockley and R. P. James, Phys. Rev. Letters **18**, 876 (1967); W. Shockley, Phys. Rev. Letters **20**, 343 (1968).

<sup>3</sup> P. Penfield and H. Haus, *The Electrodynamics of Moving Media* (M.I.T. Press, Cambridge, Mass., 1967), p. 215; Phys. Letters **26A**, 412 (1968).

<sup>4</sup> S. Coleman and J. H. Van Vleck, Phys. Rev. **171**, 1370 (1968); W. H. Furry, Amer. J. Phys. **37**, 621 (1969); the article by W. H. Furry is especially detailed and complete.

<sup>5</sup> See, for example, C. Moller, *The Theory of Relativity* (Oxford U. P., London, 1952).

<sup>6</sup> If one defines the radius vector  $\mathbf{x}_{CM}$  to the center of mass

Returning to vector notation, one sees that Eq. (30) can be rewritten as<sup>10</sup>

$$(1/c^2) (\mathcal{E} - e\phi) \psi^+ c \boldsymbol{\alpha} \psi = \frac{1}{2} \{ \psi^+ [(-i\hbar \nabla - e\mathbf{A}) \psi] + [(i\hbar \nabla - e\mathbf{A}) \psi^+] \psi \} + \nabla \times [\psi^+ \frac{1}{2} \hbar \boldsymbol{\sigma}' \psi]. \quad (36)$$

Now integrate Eq. (36) over all space. The term  $(1/c^2) \mathcal{E} \psi^+ c \boldsymbol{\alpha} \psi$  gives zero contribution since the divergence of this term is zero. The second term on the right-hand side gives the same contribution as the first term on integration by parts. The third term on the right-hand side can be rewritten as a surface integral and gives no contribution. Thus one finds

$$-(1/c^2) \int d^3 \mathbf{x} \phi \mathbf{J} = \langle \mathbf{P} \rangle_{Av}, \quad (37)$$

where  $\mathbf{J} = e \psi^+ c \boldsymbol{\alpha} \psi$  is the electric current density, and  $\langle \mathbf{P} \rangle_{Av} = \int d^3 \mathbf{x} \psi^+ (-i\hbar \nabla - e\mathbf{A}) \psi$  is the expectation value of the kinematic momentum. This is the form Eq. (11) takes in relativistic quantum mechanics.

$M \dot{\mathbf{x}}_{CM} = \int d^3 \mathbf{x} (u/c^2) \mathbf{x}$  with  $M = \int d^3 \mathbf{x} (u/c^2)$  the total mass, one can rewrite Eq. (5) as  $\mathbf{P} = M \dot{\mathbf{x}}_{CM}$ .

<sup>7</sup> Note, however, that if the current density  $\mathbf{J}$  is due to the flow of charge in conductors, then  $\mathbf{P}_{EM}$  is zero. This follows from Eq. (7) on remembering that due to induced surface charges,  $\phi$  is constant through each of the conductors, and the integral reduces to a sum of terms of the form  $\int d^3 \mathbf{x} \mathbf{J}$ , each of which according to identity (2) is zero.

<sup>8</sup> More generally one has  $\text{div} \int d^3 \mathbf{P} A \nabla f = \int d^3 \mathbf{P} [A, H] f$ .

<sup>9</sup> See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), 3rd ed., Chap. 13.

<sup>10</sup> Compare and contrast this formula with the Gordon decomposition of the current [see, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Sec. 3.3]

$$m_0 \psi^+ c \boldsymbol{\alpha} \psi = \frac{1}{2} \{ \bar{\psi} [(-i\hbar \Delta - e\mathbf{A}) \bar{\psi}] + [(i\hbar \Delta - e\mathbf{A}) \bar{\psi}] \psi \} + \nabla \times [\bar{\psi} \frac{1}{2} \hbar \boldsymbol{\sigma}' \psi] - \frac{1}{2} i \hbar c^{-1} (\partial / \partial t) (\bar{\psi} \boldsymbol{\alpha} \psi)$$

in which  $\bar{\psi} = \psi^+ \beta$ .